MAT 151 — Chapter 5

The Standard Normal Distribution

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Normal Distribution

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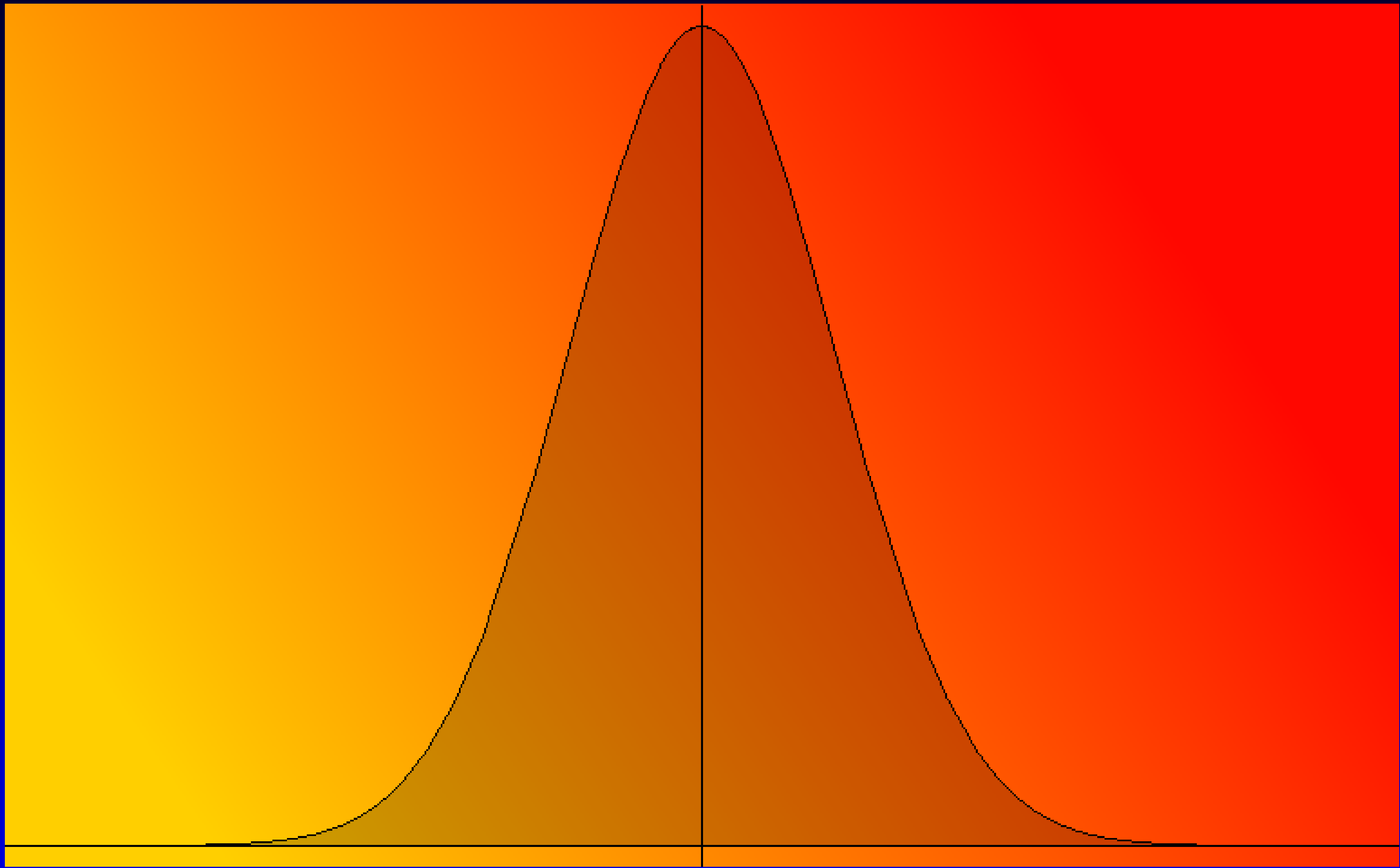
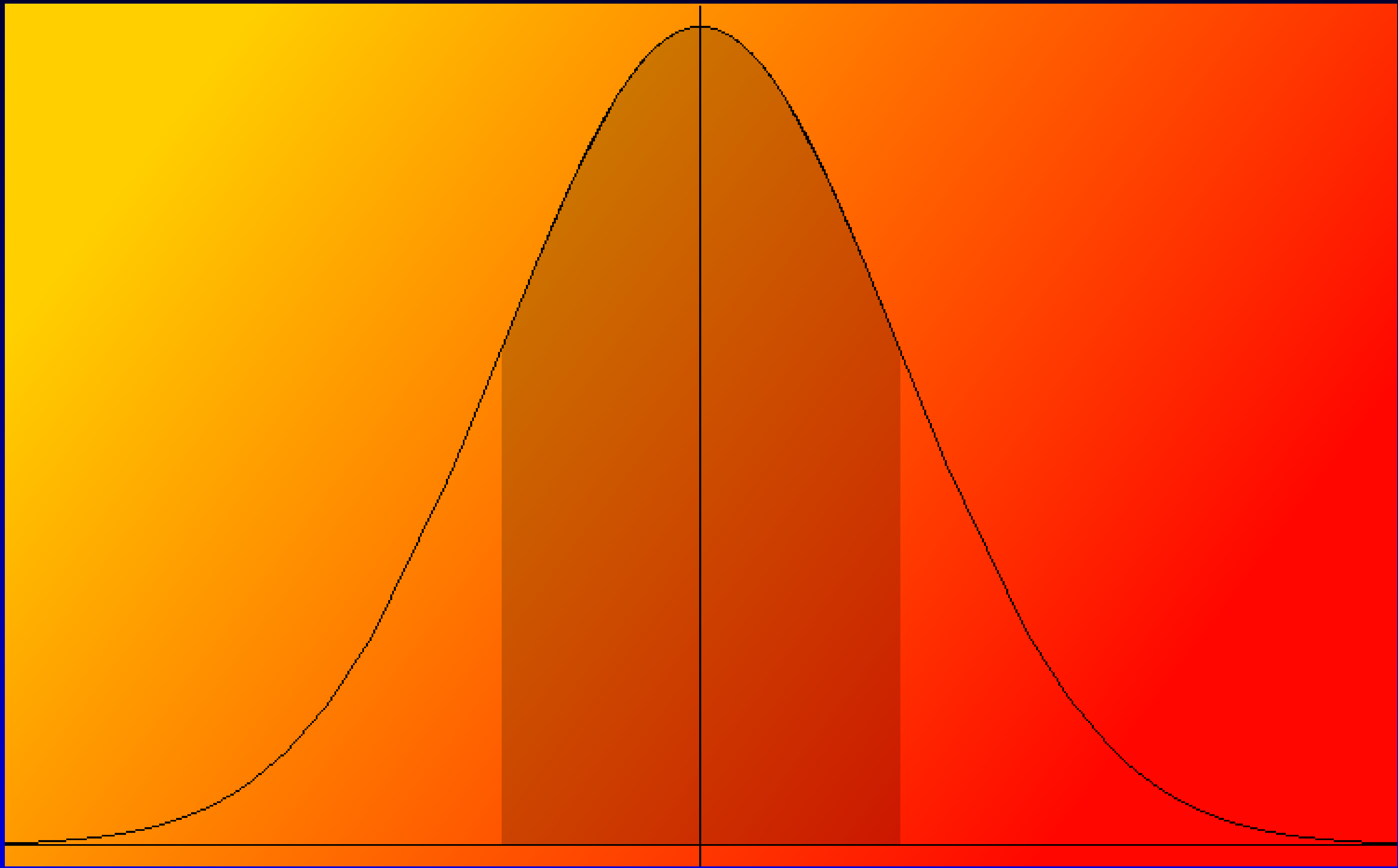
- the distribution fits the equation

$$y = \frac{e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}}{\sigma\sqrt{2\pi}}$$

Normal Distribution



Normal Distribution



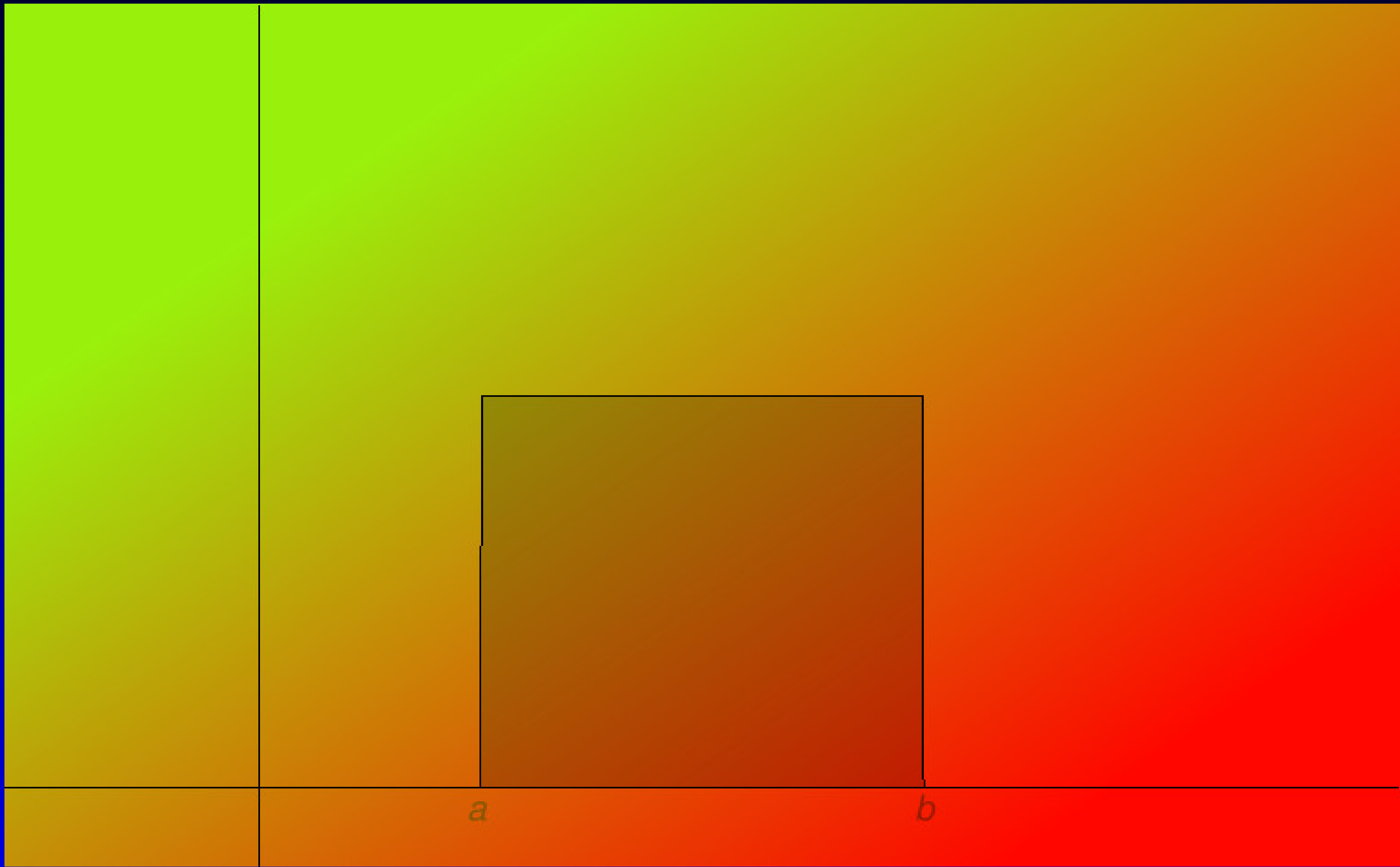
Uniform Distribution

Uniform Distribution

- A continuous random variable X has a *uniform distribution* on the interval $[a, b]$ if the distribution fits the equation

$$y = \begin{cases} \frac{1}{b-a} & \text{if } x \in [a, b] \\ 0 & \text{otherwise} \end{cases}$$

Uniform Distribution



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- It must satisfy the following properties:
  - The total area under the curve must be 1. $\left(\int_{-\infty}^{\infty} f = 1\right)$
  - $y \geq 0$ for all x.

Example

- Suppose that the temperature of a cold room is uniformly distributed from $0°$C to $5°$C.

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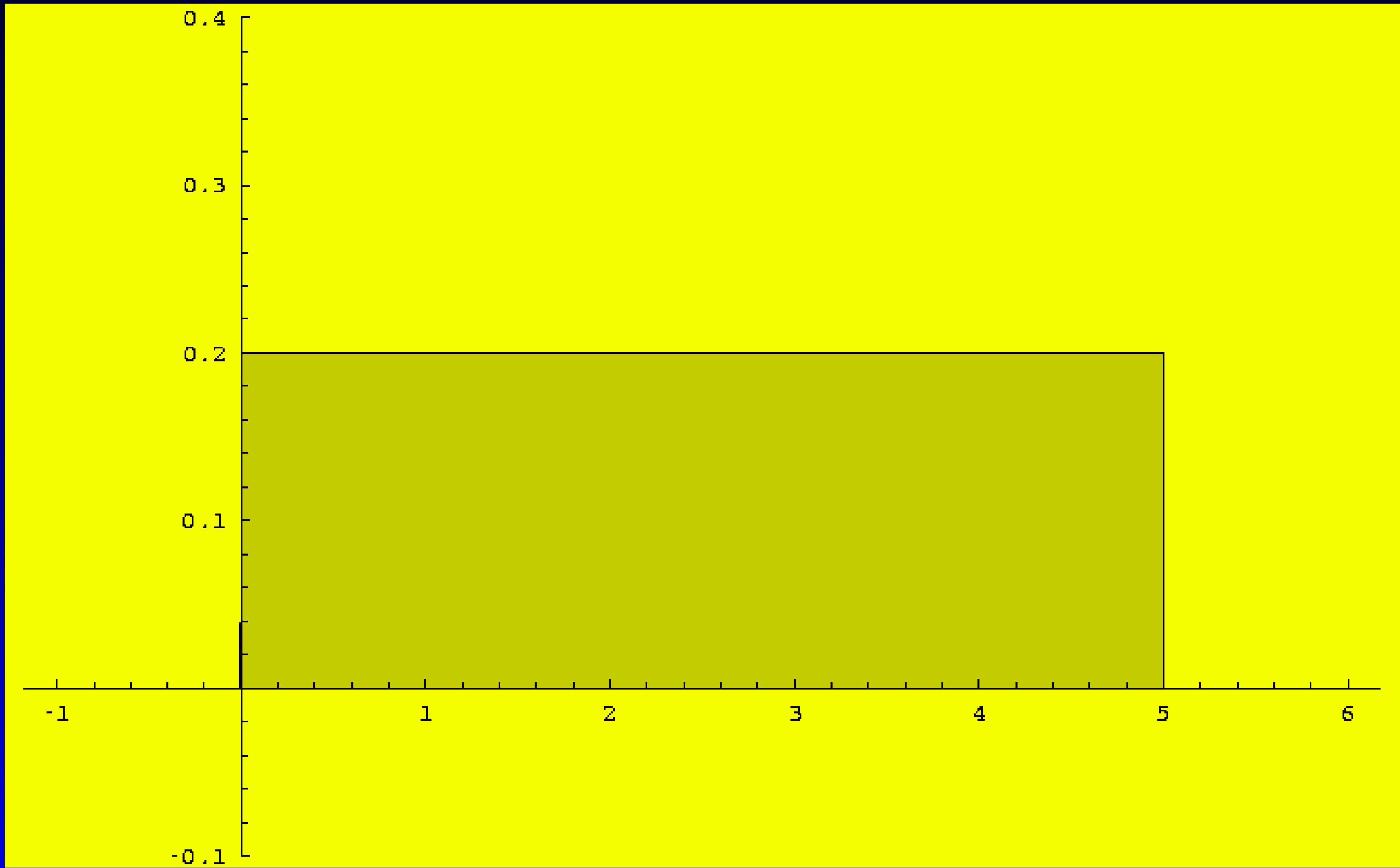
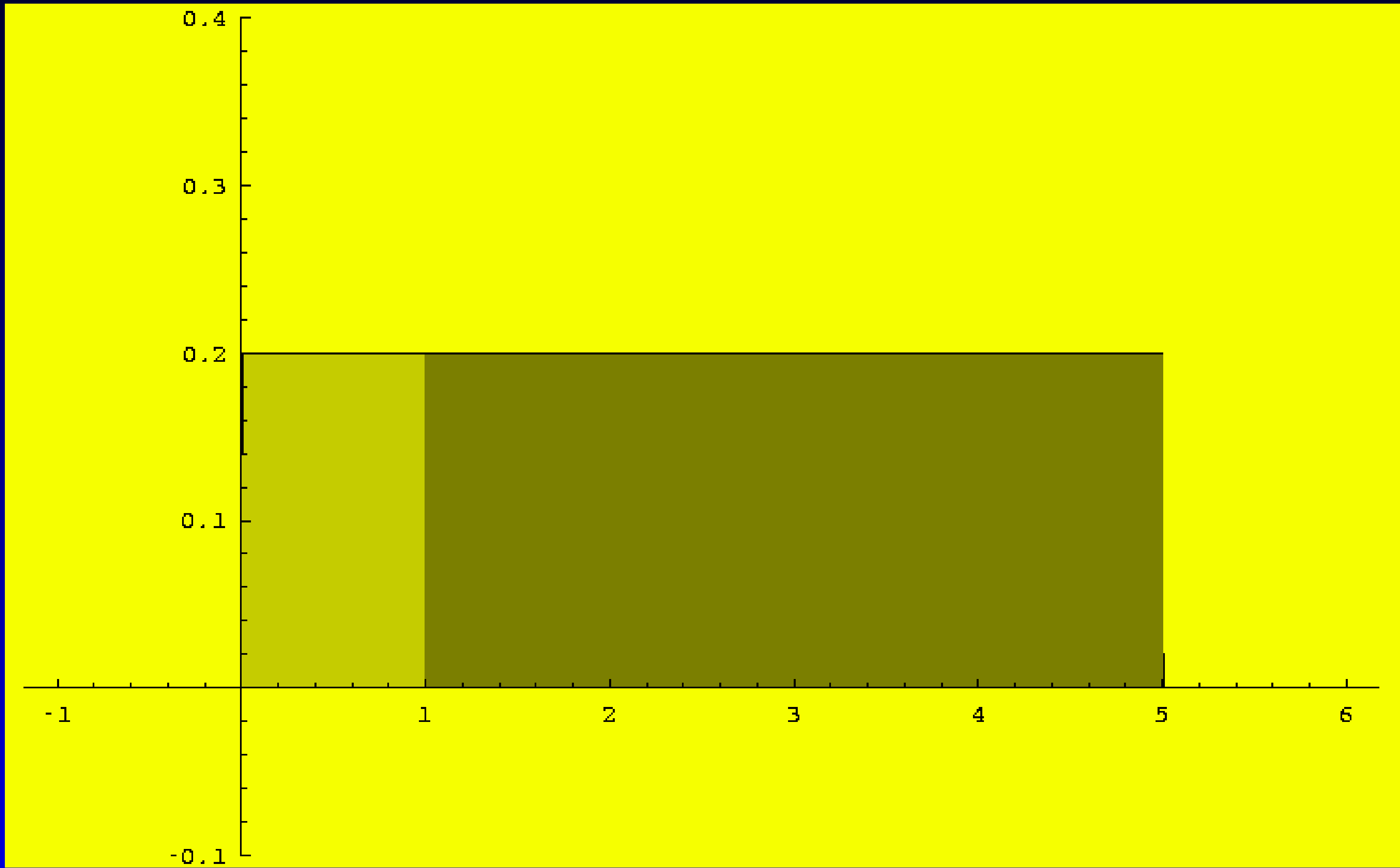
- Suppose that the temperature of a cold room is uniformly distributed from $0°$C to $5°$C.
- What is the probability that the temperature is greater than $1°$C?

Example



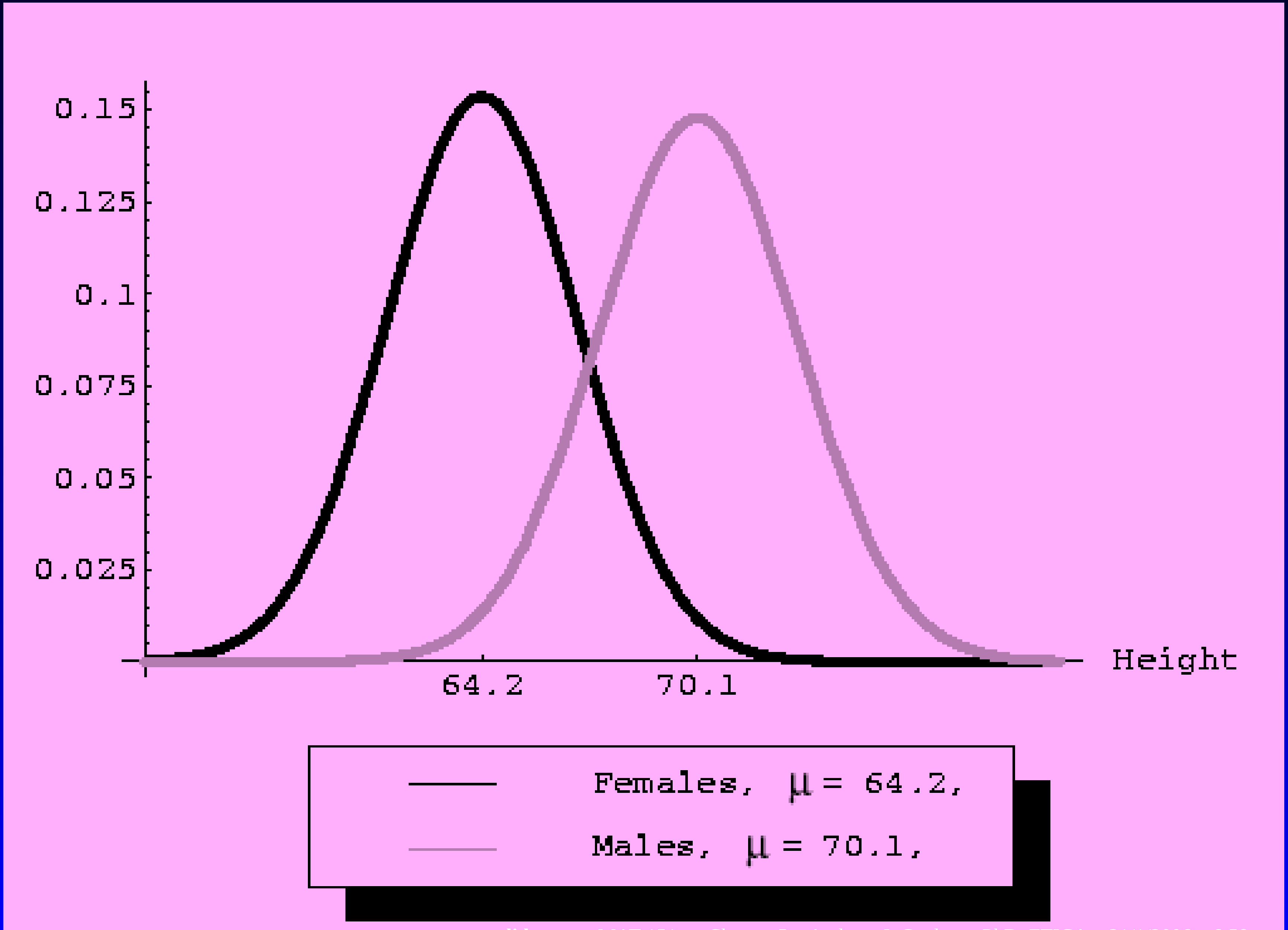
Example



Example

$$\frac{4}{5}$$

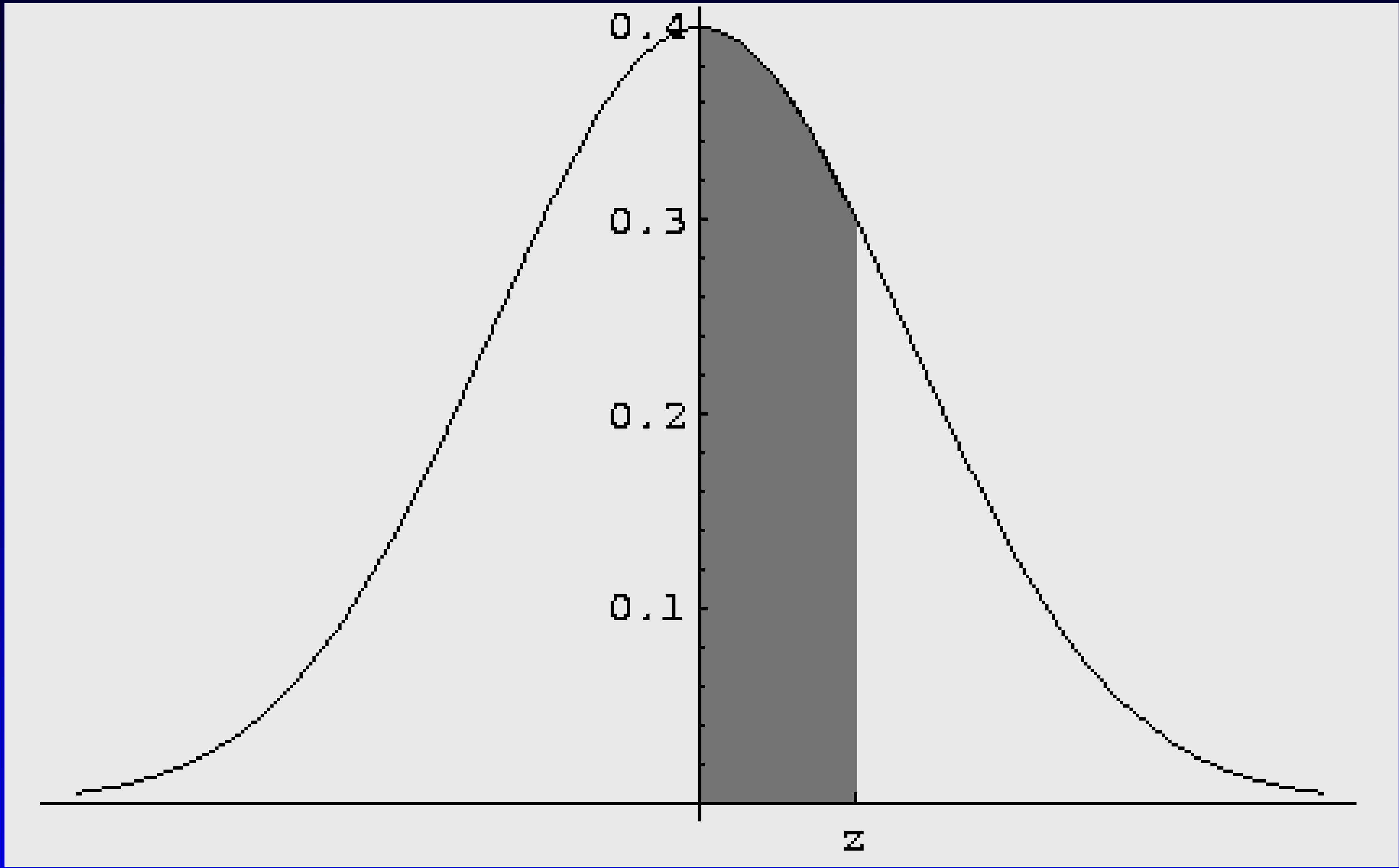
Normal Distribution



Standard Normal Distribution

The *standard normal distribution* is a normal probability distribution that has a mean of $\mu = 0$ and a standard deviation of $\sigma = 1$.

Standard Normal Distribution



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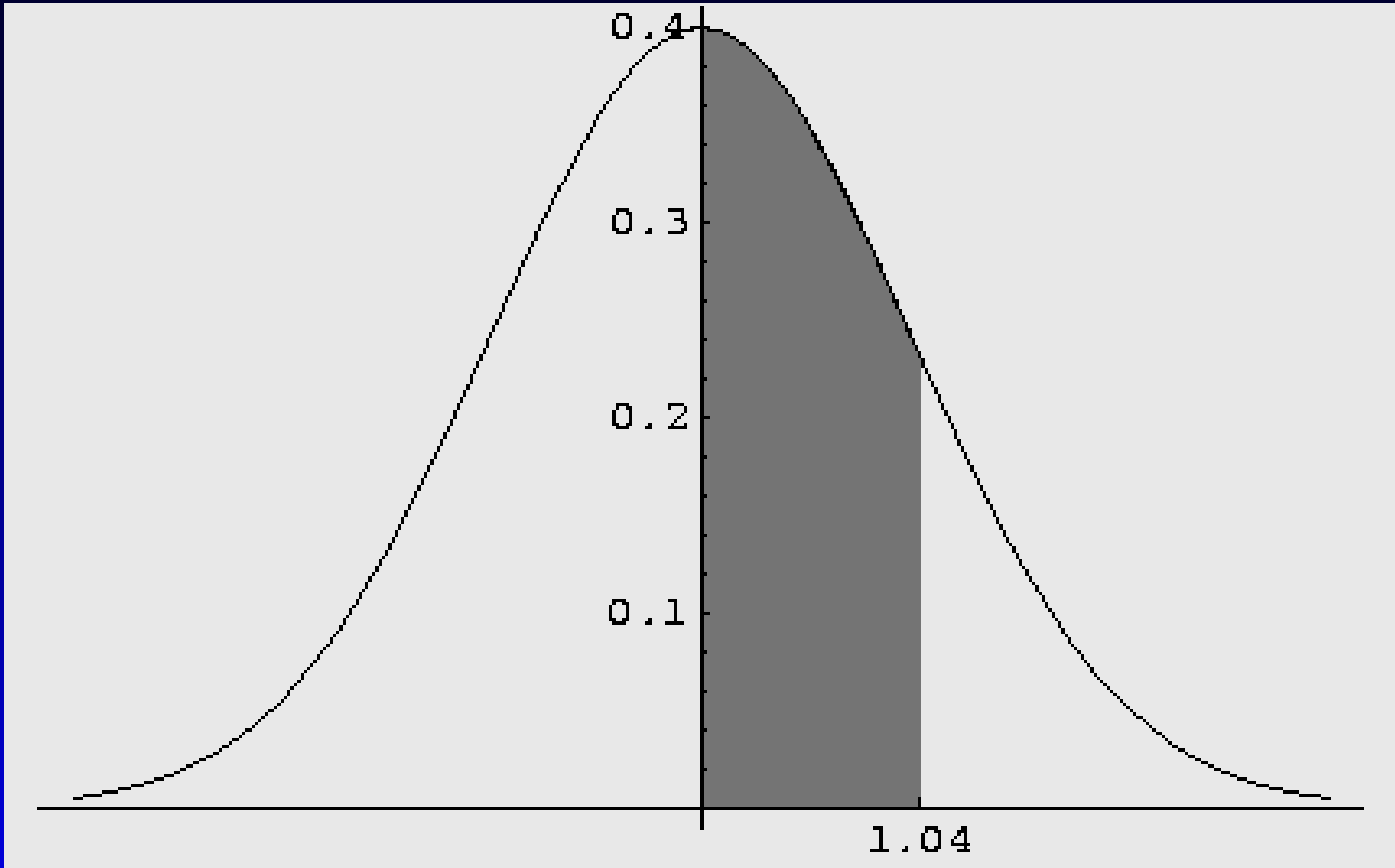
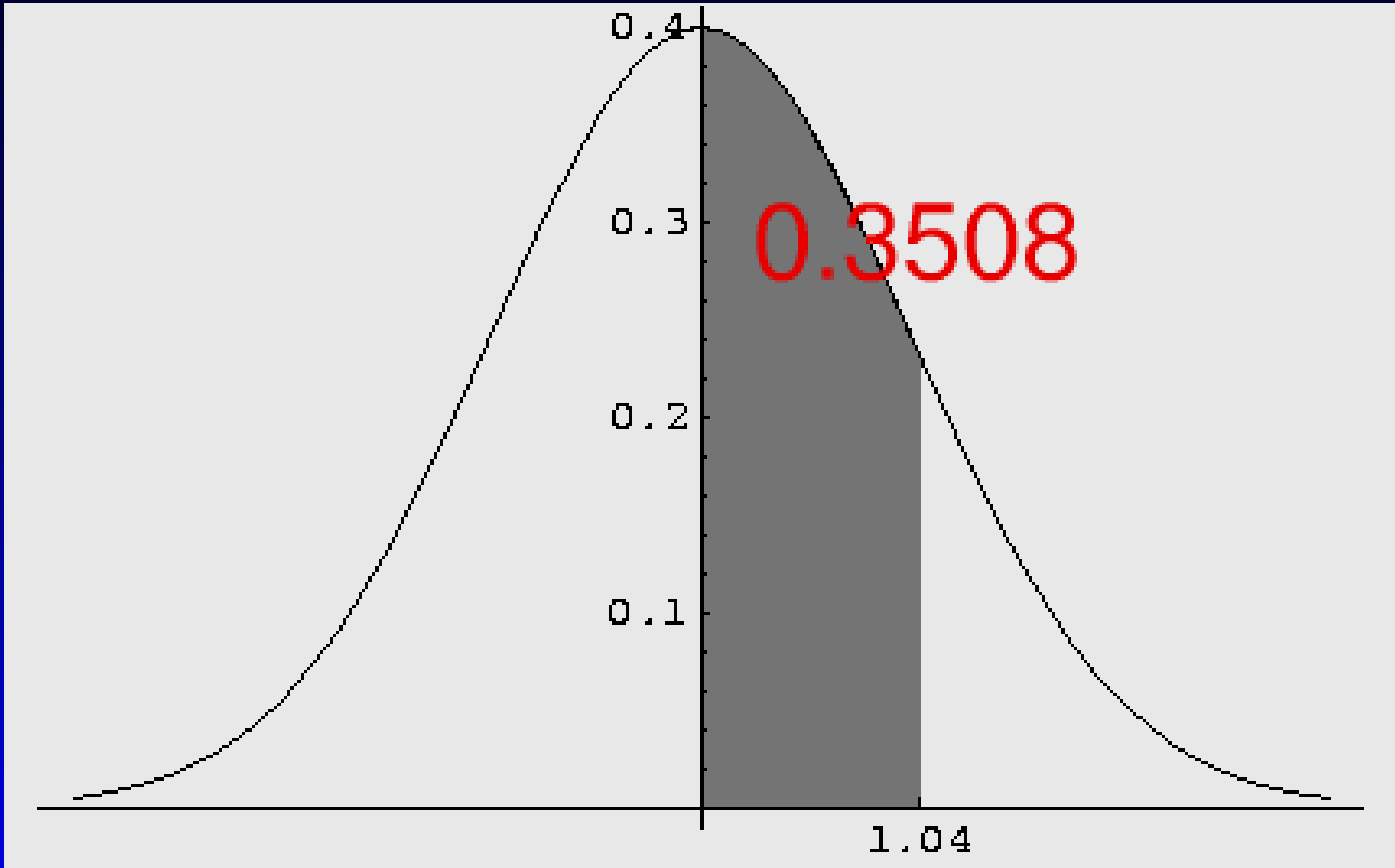


Standard Normal Distribution

TABLE A-2 Standard Normal (z) Distribution

z	.00	.01	.02	.03	.04	.05
0.0	.0000	.0040	.0080	.0120	.0160	.0199
0.1	.0398	.0438	.0478	.0517	.0557	.0596
0.2	.0793	.0832	.0871	.0910	.0948	.0987
0.3	.1179	.1217	.1255	.1293	.1331	.1368
0.4	.1554	.1591	.1628	.1664	.1700	.1736
0.5	.1915	.1950	.1985	.2019	.2054	.2088
0.6	.2257	.2291	.2324	.2357	.2389	.2422
0.7	.2580	.2611	.2642	.2673	.2704	.2734
0.8	.2881	.2910	.2939	.2967	.2995	.3023
0.9	.3159	.3186	.3212	.3238	.3264	.3289
1.0	.3413	.3438	.3461	.3485	.3508	.3531
1.1	.3643	.3665	.3686	.3708	.3729	.3749
1.2	.3849	.3869	.3888	.3907	.3925	.3944
1.3	.4032	.4049	.4066	.4082	.4099	.4115
1.4	.4192	.4207	.4222	.4236	.4251	.4265

Standard Normal Distribution



Example

The Precision Scientific Instrument Company manufactures thermometers that are supposed to give readings of $0°C$ at the freezing point of water. Tests on a large sample of these instruments reveal that at the freezing point of water some thermometers give readings below $0°C$ while others give readings above $0°C$. Assume that the mean reading is $0°C$ and the standard deviation is $1.00°C$. Also assume that the frequency distribution of errors closely resembles the normal distribution.

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If one thermometer is randomly selected, find the probability that, at the freezing point of water, the reading is between $0°$C and $+1.58°$C.

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If one thermometer is randomly selected, find the probability that, at the freezing point of water, the reading is between $-0.50°$C and $+0.50°$C.

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If one thermometer is randomly selected, find the probability that, at the freezing point of water, the reading is above $+1.60°$C.

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If one thermometer is randomly selected, find the probability that, at the freezing point of water, the reading is below $+1.00°$C.

Example

The Discount Scientific Instrument Company also manufactures thermometers that are supposed to give readings of $0°$C at the freezing point of water. Tests on a large sample of these instruments reveal that at the freezing point of water some thermometers give readings below $0°$C while others give readings above $0°$C. Assume that the mean reading is $0.1°$C but the standard deviation is $0.90°$C. Also assume that the frequency distribution of errors closely resembles the normal distribution.

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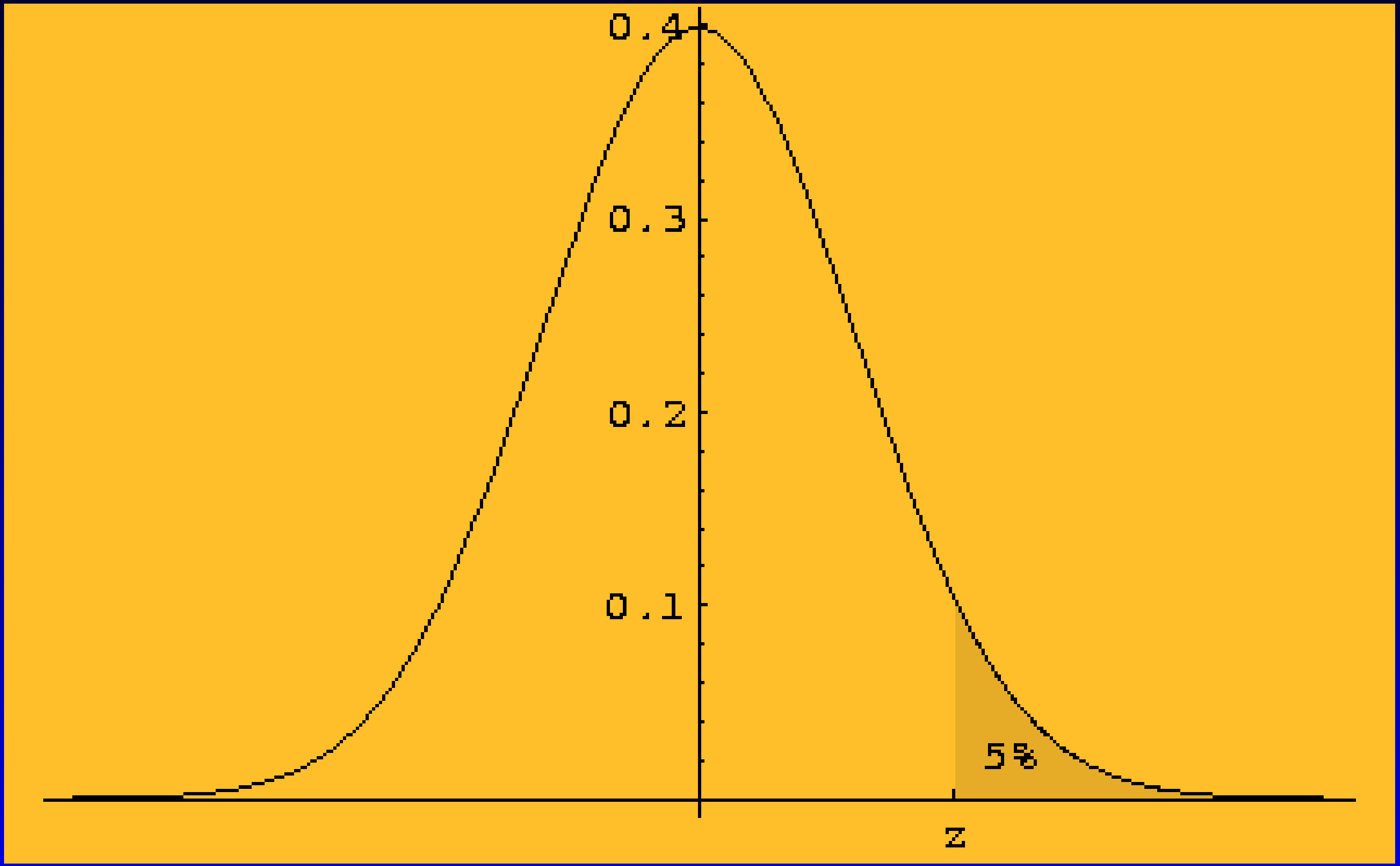
Find $P(z < 2.00)$.

Percentile

Find the 95th-percentile of the Standard Normal Distribution.

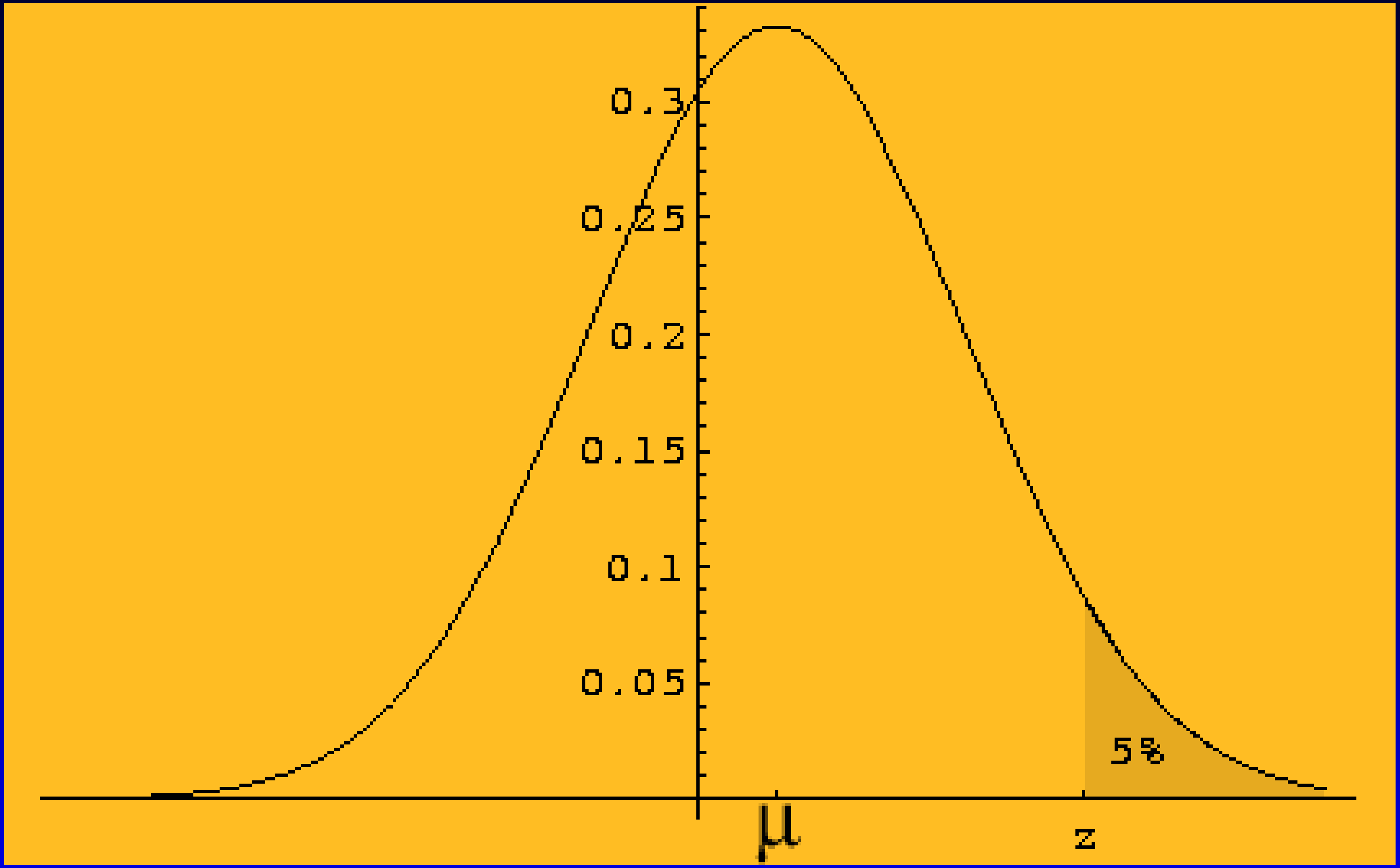
Percentile



Example

The Cheap Scientific Instrument Company is a third manufacturer of thermometers that are supposed to give readings of $0°$C at the freezing point of water. Tests on a large sample of these instruments reveal that at the freezing point of water some thermometers give readings below $0°$C while others give readings above $0°$C. Assume that the mean reading is $0.5°$C but the standard deviation is $1.20°$C. Also assume that the frequency distribution of errors closely resembles the normal distribution.
Find the 95th-percentile.

Example



Sampling Distribution

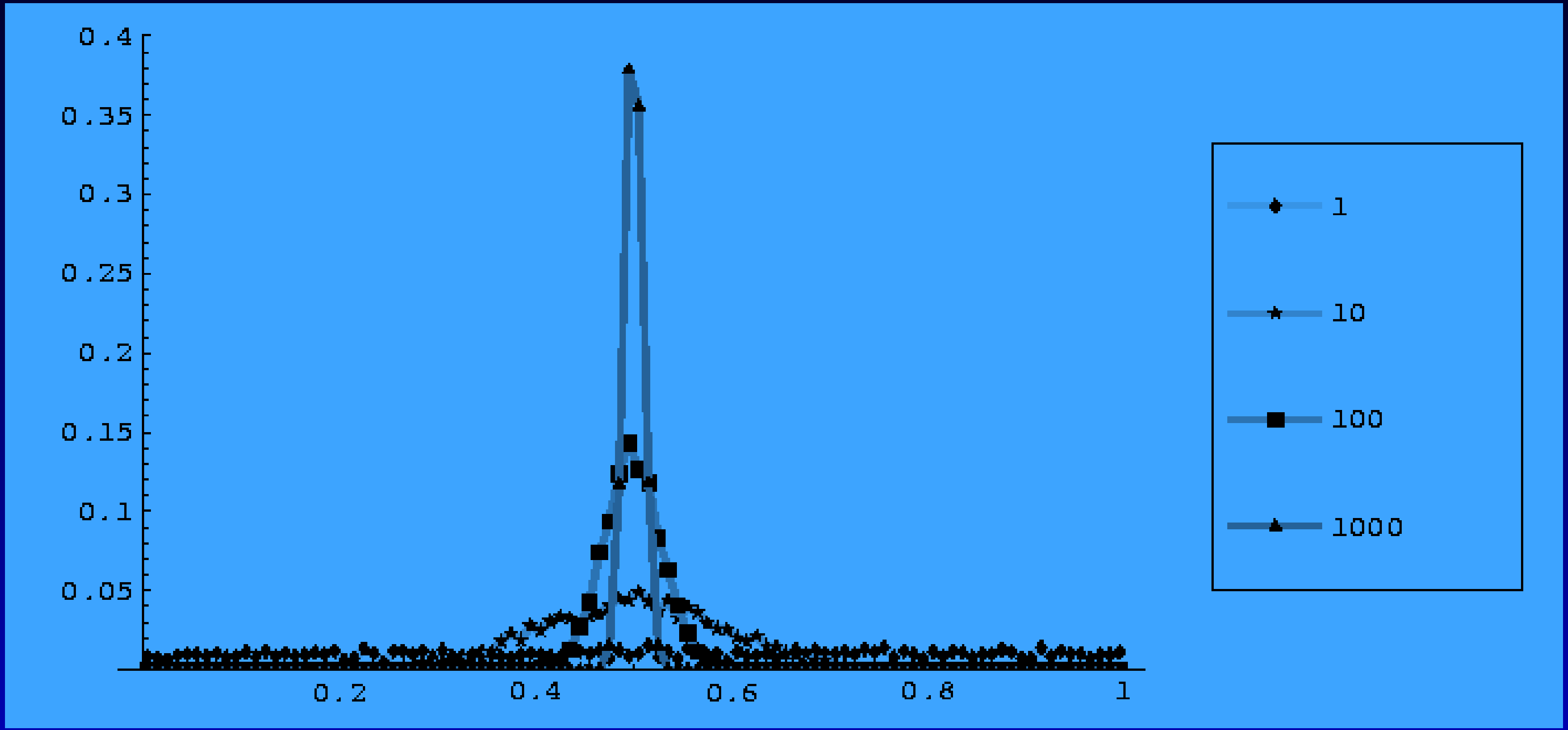
Sampling Distribution

The *sampling distribution* of the mean is the probability distribution of sample means, with all samples having the sample size n.

Simulation

Consider a random variable X.
Take 2000 samples of size n from X. Calculate $\bar{X}$ and plot the relative frequency (of 100 classes).

Simulation



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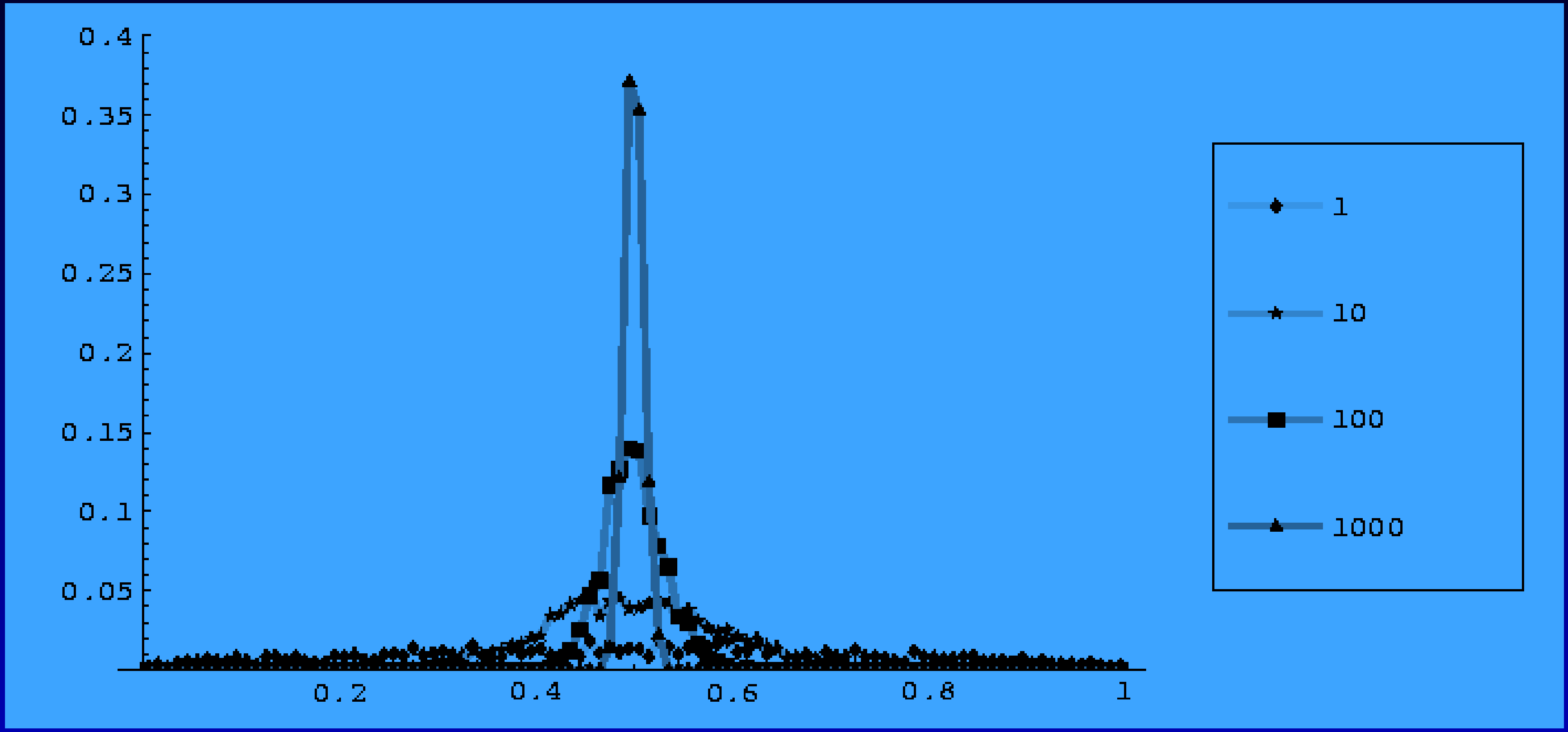

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Central Limit Theorem

When sampling without replacement and the sample size n is greater than 5% of the finite population size N, we adjust the standard deviation of sample means $\sigma_{\bar{X}}$ by multiplying it by the *finite population correction factor*

$$\sqrt{\frac{N-n}{N-1}}$$

Normal Approximation

Normal Approximation

- If $n\,p \geq 5$ and $n\,q \geq 5$ then the binomial random variable is approximately normally distributed with the mean and standard deviation given as $\mu = n\,p$ and $\sigma = \sqrt{n\,p\,q}$.

Continuity Correction

Continuity Correction

- When we use the normal distribution as an approximation to the binomial distribution, a continuity correction is made to a discrete whole number x in the binomial distribution by representing the single value x by the interval from $x - 0.5$ to $x + 0.5$.

Example

Consumer Reports (Feb. 1992) found widespread contamination of seafood in New York and Chicago supermarkets. For example, 40% of the swordfish pieces available for sale have a level of mercury above the US Food and Drug Administration (FDA) limit. Consider a random sample of 20 swordfish pieces from New York and Chicago supermarkets.

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- Use the normal approximation to the binomial distribution to calculate the probability that more than half of the 20 swordfish pieces have mercury levels exceeding the FDA limit.
- Calculate the exact probabilities.

Example

The merging process from an acceleration lane to the through lane of a freeway constitutes an important aspect of traffic operation at interchanges. A study of parallel interchange ramps in Israel revealed that many drivers do not use the entire length of parallel lanes for acceleration, but seek as soon as possible an appropriate gap in the major stream of traffic for merging (*Transportation Engineering*, Nov 1985). At one site (Yavneh), 54% of the drivers use less than half the lane length available before merging. Suppose we plan to monitor the merging patterns of a random sample of 330 drivers at the Yavneh site.

- What is the approximate probability that fewer than 100 of the drivers will use less than half the acceleration lane length before merging?

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- What is the approximate probability that fewer than 100 of the drivers will use less than half the acceleration lane length before merging?
- What is the approximate probability that 200 or more of the drivers will use less than half the acceleration lane length before merging?

Normal Probability Plot

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- A *normal probability plot* compares the original values in a set of sample data with the expected values for a sample of the given size drawn from a normally distributed population.

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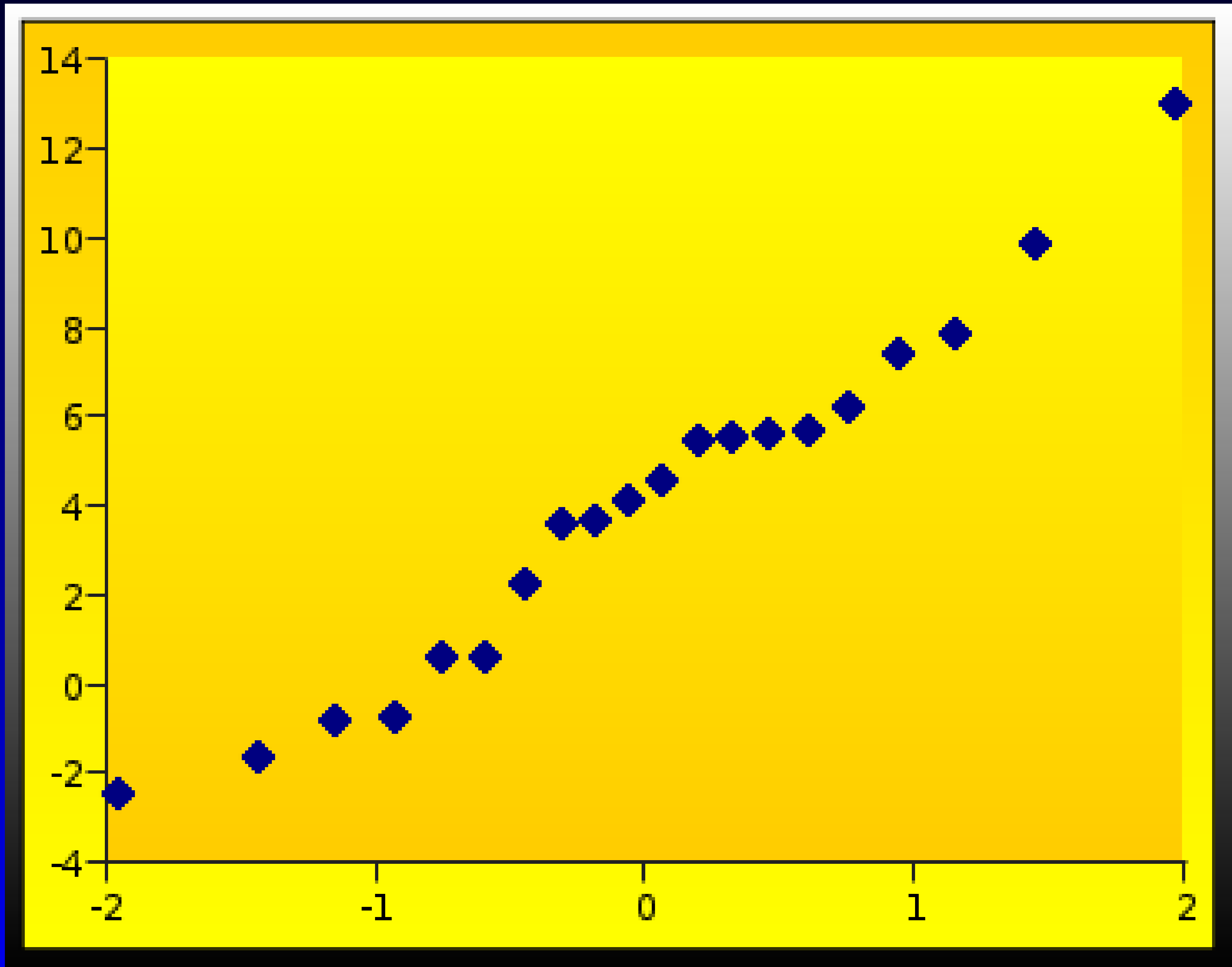
- Plot the z-scores against the observed values.
- Normally distributed data should yield an approximately straight line.

Normal Probability Plot



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