# Math 343: Homework 3

Due on February 10th, 2006

*Tyler Evans 8:00*

**Eric Anopolsky**

## p. 66 # 9

Find the number of generators of a cyclic group of order 8.

A cyclic group of order 8 is isomorphic to $\langle \mathbb{Z}_8, +_8 \rangle$ by theorem 6.10. It has 4 generators.

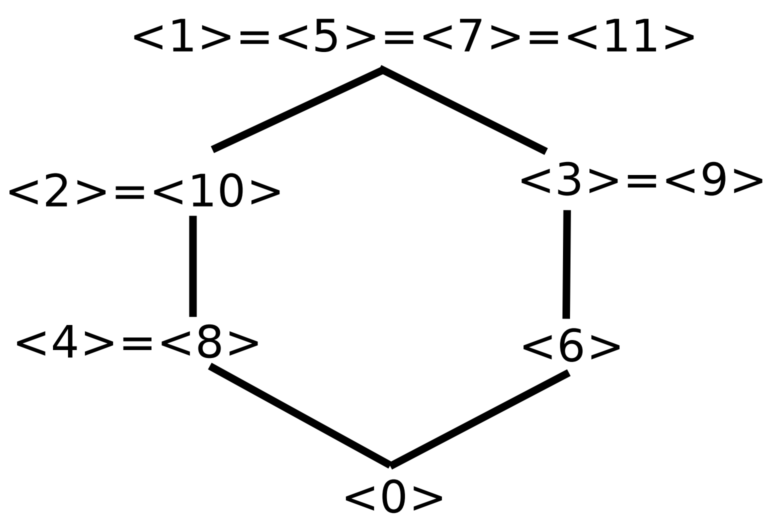
## p. 66 # 17

Find the number of elements in the cyclic subgroup of $\mathbb{Z}_{30}$ generated by 25.

The cyclic subgroup is $\{0, 5, 10, 15, 20, 25\}$, which has order 6.

## p. 66 # 22

Find all subgroups of $\mathbb{Z}_{12}$, and draw the subgroup diagram for the subgroups.




## p. 67 # 45

Let $r$ and $s$ be positive integers. Show that $\{nr + ms | n, m \in \mathbb{Z}\}$ is a subgroup of $\mathbb{Z}$.

The group is a subgroup of $\mathbb{Z}$ if it is closed under the operation, the identity element of $\mathbb{Z}$ an element, and for all $a$ in the group, $a^{-1}$ is also in the group.
Take two arbitrary elements of the group $b = n_1 r + m_1 s$ and $c = n_2 r + m_2 s$.

$$b + c = n_1 r + m_1 s + n_2 r + m_2 s = (n_1 + n_2)r + (m_1 + m_2)s,$$

which is an element of the group, so it is closed under the operation. The identity element of $\mathbb{Z}$ is $0 = -sr + rs$, which is in the group. Finally, $b^{-1} = -(n_1 r + m_1 s) = -n_1 r - m_1 s$ is also in the group, so it is a subgroup of $\mathbb{Z}$.

# p. 68 # 55

Show that $\mathbb{Z}_p$ has no proper nontrivial subgroups if $p$ is a prime number.

Let $a$ be an arbitrary element of $\mathbb{Z}_p$. If $a = \hat{0}$, $\langle a \rangle$ is the trivial subgroup. Otherwise, the cyclic subgroup generated by $a$ has order $p/\gcd(a, p)$ by theorem 6.14. Because $p$ is prime and $0 < a < p$, $\gcd(a, p)$ will always be 1, so the cyclic subgroup generated by $a$ will always have order $p$. Because it has order $p$ and is contained in $\mathbb{Z}_p$, it must be equal to $\mathbb{Z}_p$ and cannot be a proper subgroup. Therefore, no subgroup is both proper and nontrivial.