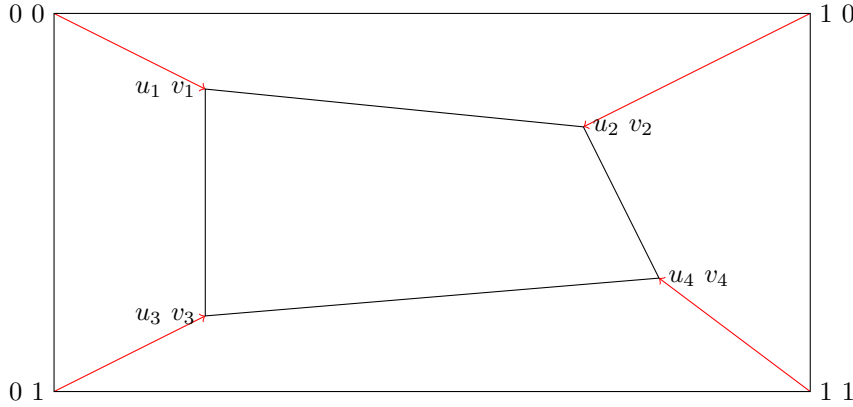# 1 Finding the transformation matrix

In homogeneous coordinates:

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} u_1 w_1 & u_2 w_2 & u_3 w_3 & u_4 w_4 \\ v_1 w_1 & v_2 w_2 & v_3 w_3 & v_4 w_4 \\ w_1 & w_2 & w_3 & w_4 \end{pmatrix}$$

$$\begin{pmatrix} a_{13} & a_{11}+a_{13} & a_{12}+a_{13} & a_{11}+a_{12}+a_{13} \\ a_{23} & a_{21}+a_{23} & a_{22}+a_{23} & a_{21}+a_{22}+a_{23} \\ 1 & a_{31}+1 & a_{32}+1 & a_{31}+a_{32}+1 \end{pmatrix} = \tag{1}$$

Identifying the coefficients of the first 3 columns:

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & 1 \end{pmatrix} = \begin{pmatrix} u_2 w_2 - u_1 & u_3 w_3 - u_1 & u_1 \\ v_2 w_2 - v_1 & v_3 w_3 - v_1 & v_1 \\ w_2 - 1 & w_3 - 1 & 1 \end{pmatrix} \tag{2}$$

Now we need w_2 and w_3...

$$\begin{aligned} w_4 &= a_{31} + a_{32} + 1 && \text{using } (1)_{34} \\ &= w_2 - 1 + w_3 - 1 + 1 && \text{using } (2)_{31}, (2)_{32} \\ &= w_2 + w_3 - 1 && (3) \end{aligned}$$

$$\begin{aligned} u_4 w_4 &= a_{11} + a_{12} + a_{13} && \text{using } (1)_{14} \\ u_4 (w_2 + w_3 - 1) &= (u_2 w_2 - u_1) + (u_3 w_3 - u_1) + u_1 && \text{using } (3), (2)_1 \\ u_4 w_2 + u_4 w_3 - u_4 &= u_2 w_2 + u_3 w_3 - u_1 \\ U_2 w_2 &= U_1 - U_3 w_3 && (4) \quad \text{with } X_i = x_i - x_4 \end{aligned}$$

$$\begin{aligned} v_4 w_4 &= a_{21} + a_{22} + a_{23} && \text{using } (1)_{24} \\ v_4 (w_2 + w_3 - 1) &= (v_2 w_2 - v_1) + (v_3 w_3 - v_1) + v_1 && \text{using } (3), (2)_2 \\ v_4 w_2 + v_4 w_3 - v_4 &= v_2 w_2 + v_3 w_3 - v_1 \\ V_3 w_3 &= V_1 - V_2 w_2 && (5) \end{aligned}$$

$$\begin{aligned} U_2 V_3 w_2 &= U_1 V_3 - U_3 V_3 w_3 && \text{using } (4) \times V_3 \\ &= U_1 V_3 - U_3 (V_1 - V_2 w_2) && \text{using } (5) \\ &= U_1 V_3 - U_3 V_1 + U_3 V_2 w_2 \\ D_{23} w_2 &= D_{13} && (6) \quad \text{with } D_{ij} = U_i V_j - U_j V_i \\ w_2 &= \frac{D_{13}}{D_{23}} && \text{if } D_{23} \neq 0 \end{aligned}$$

$$\begin{aligned} U_2 V_3 w_3 &= U_2 V_1 - U_2 V_2 w_2 && \text{using } (5) \times U_2 \\ &= U_2 V_1 - V_2 (U_1 - U_3 w_3) && \text{using } (4) \\ &= U_2 V_1 - U_1 V_2 + U_3 V_2 w_3 \\ D_{23} w_3 &= D_{21} && (7) \\ w_3 &= \frac{D_{21}}{D_{23}} && \text{if } D_{23} \neq 0 \end{aligned}$$

1.1 Degenerate case

If $D_{23} = 0$, equations 6 and 7 give $D_{13} = 0$ and $D_{21} = 0$. It means that the points p_1, p_2, p_3 and p_4 are aligned.

2 Predicting when it will blow

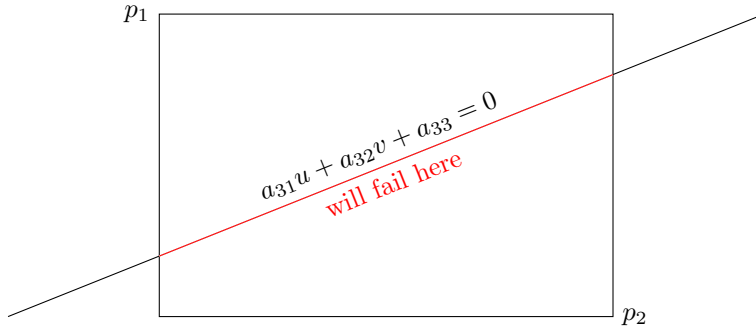### 2.1 With an obscure calculation

In homogeneous coordinates we transform:

$$\left(\begin{array}{ccc} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{array} \right) \left(\begin{array}{c} u \\ v \\ 1 \end{array} \right) = \left(\begin{array}{c} u'w' \\ v'w' \\ w' \end{array} \right) \tag{8}$$

Then we extract the transformed coordinates with:

$$\left(\begin{array}{c} u' \\ v' \end{array} \right) = \frac{1}{w'} \left(\begin{array}{c} u'w' \\ v'w' \end{array} \right)$$
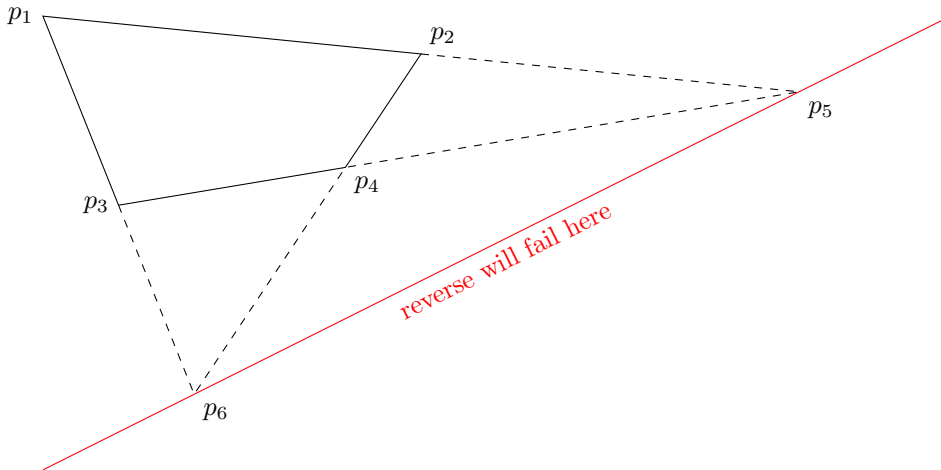
It will fail when $w' = 0$. Equation 8 gives $w' = a_{31}u + a_{32}v + a_{33}$. It means that $w' = 0$ on the line $a_{31}u + a_{32}v + a_{33} = 0$.

To be safe we need the 4 corners of the region to be on the same side of the line (all above or all below). So we check that all $a_{31}u_i + a_{32}v_j + a_{33}$ have the same sign.

2.2 With a drawing

No matter how hard we try, we will not be able to reliably reverse the perspective for the vanishing points and the line that connects them.

$$\det(p_3 - p_4, p_5 - p_4) = 0$$
$$\det(P_3, P_5) =$$
$$U_3 V_5 = U_5 V_3 \tag{9}$$

$$\det(p_5 - p_1, p_5 - p_2) = 0$$
$$\det(P_5 - P_1, P_5 - P_2) =$$
$$\det(P_5, P_5) - \det(P_1, P_5) - \det(P_5, P_2) + \det(P_1, P_2) =$$
$$\det(P_5, P_1 - P_2) = D_{21}$$
$$U_5(V_1 - V_2) - V_5(U_1 - U_2) = \tag{10}$$

$$
\begin{aligned}
U_3 D_{21} &= U_3 U_5 (V_1 - V_2) - U_3 V_5 (U_1 - U_2) && \text{using (10)} \times U_3 \\
&= U_3 U_5 (V_1 - V_2) - U_5 V_3 (U_1 - U_2) && \text{using (9)} \\
&= U_5 (U_3 V_1 - U_3 V_2 - U_1 V_3 + U_2 V_3) \\
&= U_5 (D_{23} - D_{13}) \\
U_5 &= \frac{U_3 D_{21}}{D_{23} - D_{13}} && \text{if } D_{23} \neq D_{13}
\end{aligned}
$$

$$
\begin{aligned}
V_3 D_{21} &= U_5 V_3 (V_1 - V_2) - V_3 V_5 (U_1 - U_2) && \text{using (10)} \times V_3 \\
&= U_3 V_5 (V_1 - V_2) - V_3 V_5 (U_1 - U_2) && \text{using (9)} \\
&= V_5 (U_3 V_1 - U_3 V_2 - U_1 V_3 + U_2 V_3) \\
&= V_5 (D_{23} - D_{13}) \\
V_5 &= \frac{V_3 D_{21}}{D_{23} - D_{13}} && \text{if } D_{23} \neq D_{13}
\end{aligned}
$$

The same calculation gives $U_6 = \dfrac{U_2 D_{13}}{D_{23} - D_{21}}$ and $V_6 = \dfrac{V_2 D_{13}}{D_{23} - D_{21}}$ if $D_{23} \neq D_{21}$.

2.2.1   Degenerate case

$$
\begin{aligned}
D_{23} - D_{13} &= \det(P_2 - P_1, P_3) \\
&= \det(p_2 - p_1, p_3 - p_4) \tag{11}
\end{aligned}
$$

If $D_{13} = D_{23}$, equation 11 implies that $\overrightarrow{p_1 p_2} \parallel \overrightarrow{p_3 p_4}$, so p_5 does not exist. Similarly, p_6 does not exist if $D_{21} = D_{23}$.

2.3   Equivalence of these two methods

Although I was not able to prove it formally, I checked numerically that p_5 and p_6 are on the $b_{31} u + b_{32} v + b_{33} = 0$ line, with $B := A^{-1}$. It means that the two lines found in sections 2.1 and 2.2 are one and the same. Knowing that will help me sleep at night.