In infinitesimal form this leads to the transformation law

$$\psi(x) \to \psi(x) - \imath \frac{e}{C_g} \int \frac{1}{a^d} g\left(\frac{x-b}{a}\right) f_a(b) \frac{da d^d b}{a}. \qquad (10)$$

Because of the linearity of wavelet transform the equation (9) guarantees the gauge transform $A_\mu{}'(x) = A_\mu(x) + \frac{\partial f(x)}{\partial x^\mu}$ for ordinary local fields.

Let us now specify the gauge theory and the Ward-Tahakashi identities for the theory of scale-dependent fields $A_{\mu(a)}$.

The Lagrangian itself is gauge invariant by construction and only the source term aquires a multiplication by the factor

$$\exp\left(\imath \int \left[-\frac{1}{\alpha}(\partial^\mu A_\mu)\partial^2\Lambda + J^\mu \partial_\mu \Lambda - \imath e \Lambda(\bar\eta\psi - \bar\psi\eta) \right] \right),$$

which can approximately be represented by a first order term for "small Λ":

$$1 + \imath \int dx \left[-\frac{1}{\alpha}\partial^2(\partial^\mu A_\mu) - \partial^\mu J_\mu - \imath e(\bar\eta\psi - \bar\psi\eta) \right] \Lambda(x) \equiv 1 + \imath\delta \qquad (11)$$

Let us substitute the fields by their wavelet images in those terms.

Integrating by parts we put the Laplacian $\partial^2_{x^2}$ onto the gauge fixing parameter f. Heuristically:

$$\begin{aligned} \delta &= \langle \int dx \Big[-\frac{1}{\alpha}\frac{1}{C_g^2} \int \frac{1}{a_1^d} g^\mu \left(\frac{x-b_1}{a_1}\right) A_{\mu a_1}(b_1) d1 \partial_x^2 \int \frac{1}{a_2^d} g\left(\frac{x-b_2}{a_2}\right) f_{a_2}(b_2) d2 \Big] \\ &- \frac{1}{C_g^2} \int \left[\partial^\mu J_{\mu a}(b)\right] f_a(b) \frac{da d^d b}{a} - \frac{\imath e}{C_g^3} \int \left[\bar\eta_{a_1}(b_1)\psi_{a_3}(b_3) - \bar\psi_{a_1}(b_1)\eta_{a_3}(b_3) \right] f_{a_2}(b_2) \qquad (12) \\ &\times \frac{1}{(a_1 a_2 a_3)^d} \bar g\left(\frac{x-b_1}{a_1}\right) g\left(\frac{x-b_2}{a_2}\right) g\left(\frac{x-b_3}{a_3}\right) dx d1 d2 d3 \rangle, \end{aligned}$$

where $g^\mu \equiv \frac{\partial g}{\partial x_\mu}$, $\quad d1 \equiv \frac{da_1 d^d b_1}{a_1}$. Introducing the matrix elements of operators between wavelet basic functions

$$\begin{aligned} T(1,2) &\equiv \int \frac{1}{(a_1 a_2)^d} g\left(\frac{x-b_1}{a_1}\right) \partial^2 g\left(\frac{x-b_2}{a_2}\right) dx, \\ T^{(\mu)}(1,2) &\equiv \int \frac{1}{(a_1 a_2)^d} g^\mu\left(\frac{x-b_1}{a_1}\right) \partial^2 g\left(\frac{x-b_2}{a_2}\right) dx, \qquad (13) \\ M(1,2,3) &\equiv \int \frac{1}{(a_1 a_2 a_3)^d} \bar g\left(\frac{x-b_1}{a_1}\right) g\left(\frac{x-b_2}{a_2}\right) g\left(\frac{x-b_3}{a_3}\right) dx, \end{aligned}$$

we heuristically derive the Ward-Tahakashi identities for the scale-dependent fields. In terms of above operators (13) the variation term (12) can be written in the form

$$\begin{aligned} \delta &= \langle \int -\frac{1}{\alpha}\frac{1}{C_g^2} T(1,2) \partial^\mu_{b_1} A_{\mu a_1}(b_1) f_{a_2}(b_2) d1 d2 - \frac{1}{C_g^2} \partial^\mu J_{\mu a_2}(b_2) d2 \\ &- \frac{\imath e}{C_g^3} \left[\bar\eta_{a_1}(b_1)\psi_{a_3}(b_3) - \bar\psi_{a_1}(b_1)\eta_{a_3}(b_3) \right] f_{a_2}(b_2) M(1,2,3) d1 d2 d3 \rangle \end{aligned}$$

To obtain the heuristic variation of the generating functional the fields should be substituted by corresponding functional derivatives:

$$\psi \to \frac{1}{\imath}\frac{\delta}{\delta\bar\eta}, \quad \bar\psi \to \frac{1}{\imath}\frac{\delta}{\delta\eta}, \quad A \to \frac{1}{\imath}\frac{\delta}{\delta J}$$

with all variations taken with respect to the measure on the affine group

$$dj \equiv d\mu(a_j, b_j) \equiv \frac{da_j d^d b_j}{a_j}. \qquad (14)$$

Assuming the full variation of the generating functional with respect to gauge transformations is zero, this gives the functional equation

$$\left[\frac{\imath}{\alpha}\frac{1}{C_g^2} T(1,2) \partial_1^\mu \frac{\delta}{\delta J_1^\mu} - \frac{1}{C_g^2}\partial^\mu J_{\mu 2} - \frac{e}{C_g^3}\left(\bar\eta_1 \frac{\delta}{\delta\bar\eta_3} - \eta_3 \frac{\delta}{\delta\eta_1} \right) M(1,2,3) d1 d3 \right] Z[\bar\eta, \eta, J] = 0.$$

To heuristically derive the Ward-Takahashi equations for connected Green functions we substitute

$$Z = \exp(\imath W).$$

This gives heuristic equation in functional derivatives

$$-\frac{\imath}{\alpha}\frac{1}{C_g^2} \int d1 \left[\partial_1^\mu \frac{\delta W}{\delta J_1^\mu} \right] T(1,2) - \frac{1}{C_g^2}\partial^\mu J_{\mu 2} - \frac{\imath e}{C_g^3} \int \left(\bar\eta_1 \frac{\delta W}{\delta\bar\eta_3} - \eta_3 \frac{\delta W}{\delta\eta_1} \right) M(1,2,3) d1 d3 = 0.$$

To get the equations for the vertex functions $\Gamma[\psi, \bar\psi, A_\mu]$ we apply the Legendre transform

$$\Gamma[\psi, \bar\psi, A_\mu] = W[\eta, \bar\eta, J] - \int \bar\eta\psi + \bar\psi\eta + JA \qquad (15)$$

to the latter equations. Doing so we arrive heuristically to the following equation in functional derivatives for the vertex function

$$\begin{aligned} &- \frac{1}{\alpha}\frac{1}{C_g^2} \partial_1^\mu A_\mu(1) T(1,2) + \frac{1}{C_g^2} \partial_2^\mu \frac{\delta\Gamma}{\delta A^\mu(2)} \qquad (16) \\ &- \frac{\imath e}{C_g^3} \left(\psi(3) \frac{\delta\Gamma}{\delta\psi(1)} - \bar\psi(1) \frac{\delta\Gamma}{\delta\bar\psi(3)} \right) M(1,2,3) = 0, \end{aligned}$$

where the integration over all repeated indices is assumed (14). The Ward-Takahashi equations are derived by taking the second derivatives of the equation (16) at zero fields ($A = \psi = \bar\psi = 0$). This gives

$$\begin{aligned} &\frac{1}{C_g^2} \partial_2^\mu \frac{\delta^3\Gamma[0]}{\delta\bar\psi(x_1)\delta\psi(y_1)\delta A^\mu(2)} - \frac{\imath e}{C_g^3} \frac{\delta^2}{\delta\bar\psi(x_1)\delta\psi(y_1)} \times \\ \times &\left(\psi(3) \frac{\delta\Gamma}{\delta\psi(1)} - \bar\psi(1) \frac{\delta\Gamma}{\delta\bar\psi(3)} \right) M(1,2,3) = 0, \end{aligned}$$

where the arguments of all functions are taken on the affine group, i.e. include both the resolution and the position, e.g. $x_1 \equiv (a_{x_1}, b_{x_1}), \psi(x_1) \equiv \psi_{a_{x_1}}(b_{x_1})$.

Performing the heuristical functional differentiation and using the symmetry under the permutation $2 \leftrightarrow 3$ after the integration we have the Ward-Takahashi identities

$$\begin{aligned} &- \frac{1}{C_g^2} \partial_x^\mu \frac{\delta^3\Gamma[0]}{\delta\bar\psi(x_1)\delta\psi(y_1)\delta A^\mu(x)} \qquad (17) \\ &= \imath e \frac{\delta^2\Gamma[0]}{\delta\bar\psi(1)\delta\psi(y_1)} M(x_1, x, 1) - \imath e \frac{\delta^2\Gamma[0]}{\delta\bar\psi(x_1)\delta\psi(1)} M(1, x, y_1) \end{aligned}$$

Following [8] we define vertex functions and inverse propagators in the Fourier space

$$\begin{aligned} &\int d^d b_x d^d b_{x_1} d^d b_{y_1} \exp\left(\imath(p' b_{x_1} - p b_{y_1} - q b_x)\right) \frac{\delta^3\Gamma[0]}{\delta\bar\psi(x_1)\delta\psi(y_1)\delta A^\mu(x)} \\ := \; &\imath e (2\pi)^d \delta(p' - p - q) \Gamma_{\mu a_{x_1} a_{y_1} a_x}(p, q, p') \qquad (18) \\ &\int d^d b_{x_1} d^d b_{y_1} \exp\left(\imath(p' b_{x_1} - p b_{y_1})\right) \frac{\delta^2\Gamma[0]}{\delta\bar\psi(x_1)\delta\psi(y_1)} \\ := \; &\imath (2\pi)^d \delta(p' - p) S'^{-1}_{a_{x_1} a_{y_1}}(p). \qquad (19) \end{aligned}$$

Using the definitions (18,19) we multiply equation (17) by $e^{\imath(p' b_{x_1} - p b_{y_1} - q b_x)}$ and integrate over $d^d b_x d^d b_{x_1} d^d b_{y_1}$. This gives

$$\begin{aligned} q^\mu \Gamma_{\mu a_4 a_3 a_1}(p, q, p+q) &= \int \frac{da_2}{a_2} S^{-1}_{a_1 a_2}(p+q) \tilde M_{a_2 a_3 a_4}(p+q, q, p) \qquad (20) \\ &- \int \frac{da_2}{a_2} \tilde M_{a_1 a_3 a_2}(p+q, q, p) S^{-1}_{a_2 a_4}(p), \end{aligned}$$

where

$$\tilde M_{a_1 a_2 a_3}(k_1, k_2, k_3) = (2\pi)^d \delta^d(k_1 - k_2 - k_3) \bar{\tilde g}(a_1 k_1) \tilde g(a_2 k_2) \tilde g(a_3 k_3)$$

is the Fourier image of the vertex operator (13).

The equation (20) is an exact nonlocal analog of the ordinary (local) Ward-Takahashi equation in Fourier space.

Acknowledgement

The paper was supported by DFG Project 436 RUS 113/951.

References

[1] P. Federbush. A new formulation and regularization of gauge theories using a non-linear wavelet expansion. *Progr. Theor. Phys.*, 94:1135–1146, 1995.

[2] G.V. Efimov. *Problems in quantum theory of nonlocal interactions.* Nauka, Moscow, 1985. in Russian.

[3] M.V. Altaisky. Causality and multiscale expansions in quantum field theory. *Physics of particles and nuclei letters*, 2(6):337–339, 2005.

[4] M.V. Altaisky. Wavelet-based quantum field theory. *Symmetry, Integrability and Geometry: Methods and Applications*, 3:105, 2007.

[5] V. A. Alebastrov and G. V. Efimov. Causality in quantum field theory with nonlocal interaction. *Communications in Mathematical Physics*, 38(1):11–28, 1974.

[6] J.C. Ward. An identity in quantum electrodynamics. *Phys. Rev.*, 78:182, 1950.

[7] M. V. Altaisky. Scale-dependent functions, stochastic quantization and renormalization. *Symmetry, Integrability and Geometry: Methods and Applications*, 2:046, 2006.

[8] L.H. Ryder. *Quantum field theory.* Cambridge University Press, 1985.

Gauge invariance in wavelet-based quantum field theory

S.Albeverio

Faculty of Applied Mathematics, Bonn University, IZKS, Bonn, D-53115, Germany

M.V.Altaisky

Joint Institute for Nuclear Research, Dubna, 141980, Russia*

January 19, 2009

Abstract

Continuous wavelet transform has been attracting attention as a possible tool for regularisation of gauge theories since the first paper of Feredbush [1]. However, up to the present time, this tool has been used only for identical substitution of the local fields in the local action

$$S[\phi(x)]: \quad \phi(x) \to \phi(x) = \frac{1}{C_\psi}\int \frac{1}{a^d}\psi\left(\frac{x-b}{a}\right)\phi_a(b)\frac{dad^db}{a},$$

where $\phi_a(b)$ is a field measured at point b with resolution a (by a device with aperture ψ). For the case of gauge fields this approach assumes the local gauge invariance $\phi_\mu(x) \to \phi_\mu(x) + \partial_\mu f(x)$; neither the gauge invariance of the scale-dependent fields $A_{\mu a}(x)$, nor their commutation relations have been specially treated. In present paper we consider the wavelet-based quantum field theory as a nonlocal field theory [2]. We formulate the gauge principle for the scale-dependent fields, and set up the causality relations [3, 4]. We also present the Ward-Takahashi identities for scale-dependent fields without any requirements of the final limit $a \to 0$.

1 Introduction

Troubles with ultraviolet divergences taken together with the fact that strict localisability of quantum events is just an approximation, that cannot be reached experimentally, stimulate the efforts to construct a self-consistent nonlocal field theory, at the expense of microcausality [5]. This is specially important for gauge field theories, including quantum electrodynamics and quantum chromodynamics. The former has well tested experimental consequences of the regularisation and renormalisation

*Also at Space Research Institute RAS; also at University of Dubna,Russia

results, which are nonlocal operations. The latter is still waiting for an adequate field theory valid out of the asymptotic freedom domain.

In local abelian gauge field theory the local phase transformation of the matter fields

$$\psi(x) \to e^{-\imath e f(x)}\psi(x), \quad \bar\psi(x) \to e^{\imath e f(x)}\psi(x) \tag{1}$$

is accompanied by the substitution of space-time derivatives by covariant derivatives

$$\partial_\mu \to D_\mu \equiv \partial_\mu + \imath e A_\mu, \tag{2}$$

which makes the theory invariant with respect to the local phase transformation (1), if the *gauge field* A_μ transforms accordingly:

$$A_\mu \to A_\mu + \partial_\mu f. \tag{3}$$

The generating functional of such a theory is invariant under transformations (1,3) if the source terms and gauge fixing terms are invariant, heuristically thus

$$Z[J,\bar\eta,\eta] = \mathcal{N}\int \mathcal{D}A_\mu \mathcal{D}\bar\psi\mathcal{D}\psi \exp\left(\imath\int L_{eff}dx\right), \tag{4}$$

$$L_{eff} = \imath\bar\psi\gamma^\mu D_\mu\psi - m\bar\psi\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\alpha}(\partial^\mu A_\mu)^2 + J^\mu A_\mu + \bar\eta\psi + \bar\psi\eta, \tag{5}$$

where $\mathcal{N}$ is normalisation constant, and α is a gauge fixing parameter. The constancy of the generating functional (4) under the transformations (1,3) is ensured by so-called Ward-Takahashi identities [6].

The aim of present paper is to formulate a gauge theory of the fields $A^\mu_{(a)}(x)$ that depend on both the position x and the resolution a. Same as in previous papers [7, 4] this is done by substituting the matter fields in the Lagrangian (5) in terms of their continuous wavelet transform:

$$\phi(x) = \frac{1}{C_g}\int \frac{1}{a^d}g\left(\frac{x-b}{a}\right)\phi_a(b)\frac{dad^db}{a}, x \in \mathbb{R}^d, \tag{6}$$

where C_g is a positive normalisation constant. The substitution (6) makes the local field theory (5) into a nonlocal field theory. That is why the local gauge invariance principle (1) should be reconsidered for such a theory. Using the ideas of nonlocal gauge field theory [2], we assume that the local phase invariance of the matter fields should be preserved under the substitution (6) and the gauge transformations of the scale-dependent gauge fields

$$A_{\mu(a)}(b) = \int \frac{1}{a^d}\bar g\left(\frac{x-b}{a}\right)A_\mu(x)d^dx \tag{7}$$

should be choosen accordingly to keep that invariance. This implies to the transformation conditions:

$$\psi(x) \to \psi(x)\exp\left(-\imath\frac{e}{C_g}\int\frac{1}{a^d}g\left(\frac{x-b}{a}\right)f_a(b)\frac{dad^db}{a}\right) \tag{8}$$

$$A^\mu_{(a)}(x) \to A^\mu_{(a)}(x) + \frac{\partial f_a(x)}{\partial x_\mu}. \tag{9}$$

[2] G.V. Efimov. *Problems in quantum theory of nonlocal interactions.* Nauka, Moscow, 1985. in Russian.

[3] M.V. Altaisky. Causality and multiscale expansions in quantum field theory. *Physics of particles and nuclei letters*, 2(6):337–339, 2005.

[4] M.V. Altaisky. Wavelet-based quantum field theory. *Symmetry, Integrability and Geometry: Methods and Applications*, 3:105, 2007.

[5] V. A. Alebastrov and G. V. Efimov. Causality in quantum field theory with nonlocal interaction. *Communications in Mathematical Physics*, 38(1):11–28, 1974.

[6] J.C. Ward. An identity in quantum electrodynamics. *Phys. Rev.*, 78:182, 1950.

[7] M. V. Altaisky. Scale-dependent functions, stochastic quantization and renormalization. *Symmetry, Integrability and Geometry: Methods and Applications*, 2:046, 2006.

[8] L.H. Ryder. *Quantum field theory.* Cambridge University Press, 1985.

where the arguments of all functions are taken on the affine group, i.e. include both the resolution and the position, e.g. $x_1 \equiv (a_{x_1}, b_{x_1}), \psi(x_1) \equiv \psi_{a_{x_1}}(b_{x_1})$.

Performing the heuristical functional differentiation and using the symmetry under the permutation $2 \leftrightarrow 3$ after the integration we have the Ward-Takahashi identities

$$\begin{aligned} &- \frac{1}{C_g^2}\partial_x^\mu \frac{\delta^3\Gamma[0]}{\delta\bar\psi(x_1)\delta\psi(y_1)\delta A^\mu(x)} \\ &= \imath e \frac{\delta^2\Gamma[0]}{\delta\bar\psi(1)\delta\psi(y_1)} M(x_1,x,1) - \imath e \frac{\delta^2\Gamma[0]}{\delta\bar\psi(x_1)\delta\psi(1)} M(1,x,y_1) \end{aligned} \tag{17}$$

Following [8] we define vertex functions and inverse propagators in the Fourier space

$$\begin{aligned} &\int d^db_x d^db_{x_1} d^db_{y_1} \exp\left(\imath(p'b_{x_1} - pb_{y_1} - qb_x)\right) \frac{\delta^3\Gamma[0]}{\delta\bar\psi(x_1)\delta\psi(y_1)\delta A^\mu(x)} \\ := &\ \imath e(2\pi)^d \delta(p'-p-q)\Gamma_{\mu a_{x_1} a_{y_1} a_x}(p,q,p') \end{aligned} \tag{18}$$

$$\begin{aligned} &\int d^db_{x_1} d^db_{y_1} \exp\left(\imath(p'b_{x_1} - pb_{y_1})\right) \frac{\delta^2\Gamma[0]}{\delta\bar\psi(x_1)\delta\psi(y_1)} \\ := &\ \imath(2\pi)^d \delta(p'-p) S'^{-1}_{a_{x_1} a_{y_1}}(p). \end{aligned} \tag{19}$$

Using the definitions (18,19) we multiply equation (17) by $e^{\imath(p'b_{x_1} - pb_{y_1} - qb_x)}$ and integrate over $d^db_x d^db_{x_1} d^db_{y_1}$. This gives

$$\begin{aligned} q^\mu \Gamma_{\mu a_4 a_3 a_1}(p,q,p+q) &= \int \frac{da_2}{a_2} S^{-1}_{a_1a_2}(p+q)\tilde M_{a_2a_3a_4}(p+q,q,p) \\ &- \int \frac{da_2}{a_2} \tilde M_{a_1a_3a_2}(p+q,q,p) S^{-1}_{a_2a_4}(p), \end{aligned} \tag{20}$$

where

$$\tilde M_{a_1a_2a_3}(k_1,k_2,k_3) = (2\pi)^d \delta^d(k_1-k_2-k_3)\overline{\tilde g}(a_1k_1)\tilde g(a_2k_2)\tilde g(a_3k_3)$$

is the Fourier image of the vertex operator (13).

The equation (20) is an exact nonlocal analog of the ordinary (local) Ward-Takahashi equation in Fourier space.

Acknowledgement

The paper was supported by DFG Project 436 RUS 113/951.

References

[1] P. Federbush. A new formulation and regularization of gauge theories using a non-linear wavelet expansion. *Progr. Theor. Phys.*, 94:1135–1146, 1995.

To obtain the heuristic variation of the generating functional the fields should be substituted by corresponding functional derivatives:

$$\psi \to \frac{1}{\imath}\frac{\delta}{\delta\bar\eta}, \quad \bar\psi \to \frac{1}{\imath}\frac{\delta}{\delta\eta}, \quad A \to \frac{1}{\imath}\frac{\delta}{\delta J}$$

with all variations taken with respect to the measure on the affine group

$$dj \equiv d\mu(a_j,b_j) \equiv \frac{da_j d^db_j}{a_j}. \tag{14}$$

Assuming the full variation of the generating functional with respect to gauge transformations is zero, this gives the functional equation

$$\left[\frac{\imath}{\alpha}\frac{1}{C_g^2}T(1,2)\partial_1^\mu \frac{\delta}{\delta J_1^\mu} - \frac{1}{C_g^2}\partial^\mu J_{\mu 2} - \frac{e}{C_g^3}\left(\bar\eta_1 \frac{\delta}{\delta\bar\eta_3} - \eta_3\frac{\delta}{\delta\eta_1}\right) M(1,2,3)d1d3\right] Z[\bar\eta,\eta,J] = 0.$$

To heuristically derive the Ward-Takahashi equations for connected Green functions we substitute

$$Z = \exp(\imath W).$$

This gives heuristic equation in functional derivatives

$$-\frac{\imath}{\alpha}\frac{1}{C_g^2}\int d1 \left[\partial_1^\mu \frac{\delta W}{\delta J_1^\mu}\right] T(1,2) - \frac{1}{C_g^2}\partial^\mu J_{\mu 2} - \frac{\imath e}{C_g^3}\int \left(\bar\eta_1 \frac{\delta W}{\delta\bar\eta_3} - \eta_3 \frac{\delta W}{\delta \eta_1}\right) M(1,2,3)d1d3 = 0.$$

To get the equations for the vertex functions $\Gamma[\psi,\bar\psi,A_\mu]$ we apply the Legendre transform

$$\Gamma[\psi,\bar\psi,A_\mu] = W[\eta,\bar\eta,J] - \int \bar\eta\psi + \bar\psi\eta + JA \tag{15}$$

to the latter equations. Doing so we arrive heuristically to the following equation in functional derivatives for the vertex function

$$\begin{aligned} &- \frac{1}{\alpha}\frac{1}{C_g^2}\partial_1^\mu A_\mu(1)T(1,2) + \frac{1}{C_g^2}\partial_2^\mu \frac{\delta\Gamma}{\delta A^\mu(2)} \\ &- \frac{\imath e}{C_g^3}\left(\psi(3)\frac{\delta\Gamma}{\delta\psi(1)} - \bar\psi(1)\frac{\delta\Gamma}{\delta\bar\psi(3)}\right) M(1,2,3) = 0, \end{aligned} \tag{16}$$

where the integration over all repeated indices is assumed (14). The Ward-Takahashi equations are derived by taking the second derivatives of the equation (16) at zero fields ($A = \psi = \bar\psi = 0$). This gives

$$\begin{aligned} &\frac{1}{C_g^2}\partial_2^\mu \frac{\delta^3\Gamma[0]}{\delta\bar\psi(x_1)\delta\psi(y_1)\delta A^\mu(2)} - \frac{\imath e}{C_g^3}\frac{\delta^2}{\delta\bar\psi(x_1)\delta\psi(y_1)} \times \\ \times &\left(\psi(3)\frac{\delta\Gamma}{\delta\psi(1)} - \bar\psi(1)\frac{\delta\Gamma}{\delta\bar\psi(3)}\right) M(1,2,3) = 0, \end{aligned}$$

In infinitesimal form this leads to the transformation law

$$\psi(x) \to \psi(x) - \imath\frac{e}{C_g}\int \frac{1}{a^d} g\left(\frac{x-b}{a}\right) f_a(b)\frac{dad^db}{a}. \tag{10}$$

Because of the linearity of wavelet transform the equation (9) guarantees the gauge transform $A_\mu{}'(x) = A_\mu(x) + \frac{\partial f(x)}{\partial x^\mu}$ for ordinary local fields.

Let us now specify the gauge theory and the Ward-Tahakashi identities for the theory of scale-dependent fields $A_{\mu(a)}$.

The Lagrangian itself is gauge invariant by construction and only the source term aquires a multiplication by the factor

$$\exp\left(\imath\int\left[-\frac{1}{\alpha}(\partial^\mu A_\mu)\partial^2\Lambda + J^\mu\partial_\mu\Lambda - \imath e\Lambda(\bar\eta\psi - \bar\psi\eta)\right]\right),$$

which can approximately be represented by a first order term for “small Λ”:

$$1 + \imath\int dx \left[-\frac{1}{\alpha}\partial^2(\partial^\mu A_\mu) - \partial^\mu J_\mu - \imath e(\bar\eta\psi - \bar\psi\eta)\right]\Lambda(x) \equiv 1 + \imath\delta \tag{11}$$

Let us substitute the fields by their wavelet images in those terms.

Integrating by parts we put the Laplacian $\partial^2_{x^2}$ onto the gauge fixing parameter f. Heuristically:

$$\begin{aligned} \delta &= \langle\int dx\Big[-\frac{1}{\alpha}\frac{1}{C_g^2}\int\frac{1}{a_1^d}g^\mu\left(\frac{x-b_1}{a_1}\right)A_{\mu a_1}(b_1)d1\partial_x^2\int\frac{1}{a_2^d}g\left(\frac{x-b_2}{a_2}\right)f_{a_2}(b_2)d2\Big] \\ &- \frac{1}{C_g^2}\int\left[\partial^\mu J_{\mu a}(b)\right]f_a(b)\frac{dad^db}{a} - \frac{\imath e}{C_g^3}\int\left[\bar\eta_{a_1}(b_1)\psi_{a_3}(b_3) - \bar\psi_{a_1}(b_1)\eta_{a_3}(b_3)\right]f_{a_2}(b_2) \\ &\times \frac{1}{(a_1a_2a_3)^d}\bar g\left(\frac{x-b_1}{a_1}\right)g\left(\frac{x-b_2}{a_2}\right)g\left(\frac{x-b_3}{a_3}\right)dxd1d2d3\rangle, \end{aligned} \tag{12}$$

where $g^\mu \equiv \frac{\partial g}{\partial x_\mu}$, $\quad d1 \equiv \frac{da_1d^db_1}{a_1}$. Introducing the matrix elements of operators between wavelet basic functions

$$\begin{aligned} T(1,2) &\equiv \int\frac{1}{(a_1a_2)^d}g\left(\frac{x-b_1}{a_1}\right)\partial^2 g\left(\frac{x-b_2}{a_2}\right)dx, \\ T^{(\mu)}(1,2) &\equiv \int\frac{1}{(a_1a_2)^d}g^\mu\left(\frac{x-b_1}{a_1}\right)\partial^2 g\left(\frac{x-b_2}{a_2}\right)dx, \\ M(1,2,3) &\equiv \int\frac{1}{(a_1a_2a_3)^d}\bar g\left(\frac{x-b_1}{a_1}\right)g\left(\frac{x-b_2}{a_2}\right)g\left(\frac{x-b_3}{a_3}\right)dx, \end{aligned} \tag{13}$$

we heuristically derive the Ward-Tahakashi identities for the scale-dependent fields. In terms of above operators (13) the variation term (12) can be written in the form

$$\begin{aligned} \delta &= \langle\int -\frac{1}{\alpha}\frac{1}{C_g^2}T(1,2)\partial^\mu_{b_1}A_{\mu a_1}(b_1)f_{a_2}(b_2)d1d2 - \frac{1}{C_g^2}\partial^\mu J_{\mu a_2}(b_2)d2 \\ &- \frac{\imath e}{C_g^3}\left[\bar\eta_{a_1}(b_1)\psi_{a_3}(b_3) - \bar\psi_{a_1}(b_1)\eta_{a_3}(b_3)\right]f_{a_2}(b_2)M(1,2,3)d1d2d3\rangle \end{aligned}$$

Gauge invariance in wavelet-based quantum field theory

S.Albeverio

Faculty of Applied Mathematics, Bonn University, IZKS, Bonn, D-53115, Germany

M.V.Altaisky

Joint Institute for Nuclear Research, Dubna, 141980, Russia*

January 19, 2009

Abstract

Continuous wavelet transform has been attracting attention as a possible tool for regularisation of gauge theories since the first paper of Feredbush [1]. However, up to the present time, this tool has been used only for identical substitution of the local fields in the local action

$$S[\phi(x)]: \quad \phi(x) \to \phi(x) = \frac{1}{C_\psi}\int \frac{1}{a^d}\psi\left(\frac{x-b}{a}\right)\phi_a(b)\frac{dad^db}{a},$$

where $\phi_a(b)$ is a field measured at point b with resolution a (by a device with aperture ψ). For the case of gauge fields this approach assumes the local gauge invariance $\phi_\mu(x) \to \phi_\mu(x) + \partial_\mu f(x)$; neither the gauge invariance of the scale-dependent fields $A_{\mu a}(x)$, nor their commutation relations have been specially treated. In present paper we consider the wavelet-based quantum field theory as a nonlocal field theory [2]. We formulate the gauge principle for the scale-dependent fields, and set up the causality relations [3, 4]. We also present the Ward-Takahashi identities for scale-dependent fields without any requirements of the final limit $a \to 0$.

1 Introduction

Troubles with ultraviolet divergences taken together with the fact that strict localisability of quantum events is just an approximation, that cannot be reached experimentally, stimulate the efforts to construct a self-consistent nonlocal field theory, at the expense of microcausality [5]. This is specially important for gauge field theories, including quantum electrodynamics and quantum chromodynamics. The former has well tested experimental consequences of the regularisation and renormalisation

*Also at Space Research Institute RAS; also at University of Dubna,Russia

results, which are nonlocal operations. The latter is still waiting for an adequate field theory valid out of the asymptotic freedom domain.

In local abelian gauge field theory the local phase transformation of the matter fields

$$\psi(x) \to e^{-\imath e f(x)}\psi(x), \quad \bar\psi(x) \to e^{\imath e f(x)}\psi(x) \tag{1}$$

is accompanied by the substitution of space-time derivatives by covariant derivatives

$$\partial_\mu \to D_\mu \equiv \partial_\mu + \imath e A_\mu, \tag{2}$$

which makes the theory invariant with respect to the local phase transformation (1), if the *gauge field* A_μ transforms accordingly:

$$A_\mu \to A_\mu + \partial_\mu f. \tag{3}$$

The generating functional of such a theory is invariant under transformations (1,3) if the source terms and gauge fixing terms are invariant, heuristically thus

$$Z[J,\bar\eta,\eta] = \mathcal{N}\int \mathcal{D}A_\mu \mathcal{D}\bar\psi\mathcal{D}\psi \exp\left(\imath\int L_{eff}dx\right), \tag{4}$$

$$L_{eff} = \imath\bar\psi\gamma^\mu D_\mu\psi - m\bar\psi\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\alpha}(\partial^\mu A_\mu)^2 + J^\mu A_\mu + \bar\eta\psi + \bar\psi\eta, \tag{5}$$

where $\mathcal{N}$ is normalisation constant, and α is a gauge fixing parameter. The constancy of the generating functional (4) under the transformations (1,3) is ensured by so-called Ward-Takahashi identities [6].

The aim of present paper is to formulate a gauge theory of the fields $A^\mu_{(a)}(x)$ that depend on both the position x and the resolution a. Same as in previous papers [7, 4] this is done by substituting the matter fields in the Lagrangian (5) in terms of their continuous wavelet transform:

$$\phi(x) = \frac{1}{C_g}\int \frac{1}{a^d}g\left(\frac{x-b}{a}\right)\phi_a(b)\frac{dad^db}{a}, x \in \mathbb{R}^d, \tag{6}$$

where C_g is a positive normalisation constant. The substitution (6) makes the local field theory (5) into a nonlocal field theory. That is why the local gauge invariance principle (1) should be reconsidered for such a theory. Using the ideas of nonlocal gauge field theory [2], we assume that the local phase invariance of the matter fields should be preserved under the substitution (6) and the gauge transformations of the scale-dependent gauge fields

$$A_{\mu(a)}(b) = \int \frac{1}{a^d}\bar g\left(\frac{x-b}{a}\right)A_\mu(x)d^dx \tag{7}$$

should be choosen accordingly to keep that invariance. This implies to the transformation conditions:

$$\psi(x) \to \psi(x)\exp\left(-\imath\frac{e}{C_g}\int\frac{1}{a^d}g\left(\frac{x-b}{a}\right)f_a(b)\frac{dad^db}{a}\right) \tag{8}$$

$$A^\mu_{(a)}(x) \to A^\mu_{(a)}(x) + \frac{\partial f_a(x)}{\partial x_\mu}. \tag{9}$$