15-251: Great Theoretical Ideas **Guru:** Anton Bachin

Assignment 8 Due: Thursday, March 23

0 Warmup (0 points)

This is a warmup problem. It is intended to help you practice problems of this kind. When you are done, you may look at the solutions posted on the course calendar.

Recall that a graph is said to be *k-colorable* if there is some way to assign exactly one of k colors to each vertex of the graph such that if two vertices are adjacent, then they are of different colors. Note that not all k colors must be used. Show that every tree is 2-colorable.

1 Handshaking (10 points)

Six people attend a party. Perhaps not surprisingly, some handshaking takes place. Show that at least one of the following must be true:

- There are three people at the party that all shook hands with each other.
- There are three people at the party such that none of them shook hands with each other.

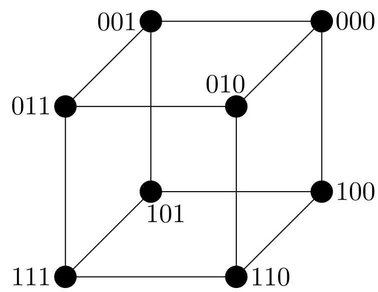
2 Coloring a Planar Graph (20 points)

In class, we proved *Euler's theorem* for planar graphs. The theorem states that if a connected simple planar graph has n vertices, f faces, and e edges, then $n - e + f = 2$. You may assume that any graphs mentioned are connected. Here is a nice application of Euler's theorem:

- **(4 points)** Let G be a simple planar graph on n vertices with e edges (recall that a graph is simple if it has no parallel edges or loops). Show that if $n \geq 3$, then $e \leq 3n - 6$.
- **(2 points)** Let G be a simple planar graph. Show that there is some vertex in G with degree at most 5.
- **(4 points)** Use this fact to prove by induction that every simple planar graph is 6-colorable.
- **(10 points)** Turn your proof into an algorithm that colors the vertices of any simple planar graph using at most 6 colors.

3 Hypercube Network (25 points)

Consider the following: take all binary numbers of length k and make a vertex for each; call the binary number corresponding to each vertex the label of the vertex. Join two vertices with an edge if their labels differ in only one bit. The resulting graph is called a hypercube. Below is an example for $k = 3$ (the 3-cube):



A *matching* is a graph in which pairs of vertices are joined by edges, and every vertex is paired with exactly one other vertex (each vertex has degree one). A *perfect matching* of a graph G is a subgraph of G that is a matching and includes all the vertices of G.

- **(5 points)** To begin, show that for $k > 0$, the k-cube has a perfect matching.

In lecture, you briefly saw the pancake graph, which has interesting connectivity properties. In the rest of this problem, you will investigate the connectivity properties of hypercubes.

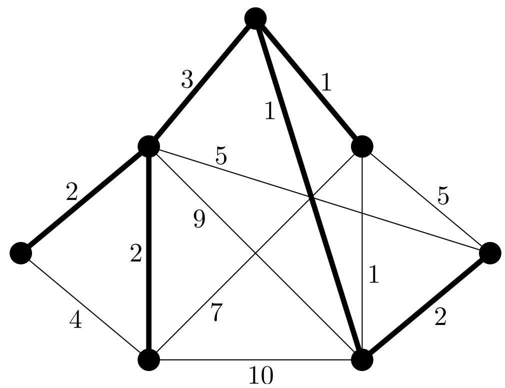
- **(5 points)** Find and prove a closed form expression for the number of edges in the k-cube (hint: it might be useful to search for an inductive definition of the k-cube, i.e. a procedure for making the $(k + 1)$-cube from the k-cube).
- **(15 points)** Determine the minimum number of edges that must be removed to disconnect the k-cube, and prove your result. In other words, find some number $f(k)$ such that if fewer than $f(k)$ edges are removed from the k-cube, it must remain connected, and show that there is some set of $f(k)$ edges whose removal will disconnect the k-cube.

4 Spanning Tree (15 points)

A *weighted graph* is a graph in which each edge has a number, called a *weight*, associated with it. This weight may be used to represent, for example, a distance, if the vertices of an edge represent cities. The weight of a graph is the sum of the weights of all its edges.

Recall that a *spanning tree* of a connected graph G is a subgraph that is a tree and includes all the vertices of G. *A minimum-weight spanning tree* of a weighted connected graph G, then, is a spanning tree of G such that no other spanning tree of G has lesser weight.

Here is an example of a mimimum-weight spanning tree, highlighted in bold in a weighted graph:



Show that if the edges of a weighted graph have unique weights, the graph has a unique minimum-weight spanning tree.

5 Unicycles (10 points)

Given a graph G, a simple cycle of length l in G is a sequence of vertices $v_0, \ldots, v_{l-1}$, in which no vertex is repeated, $l \geq 3$, and v_i is adjacent to both $v_{i-1 \bmod l}$ and $v_{i+1 \bmod l}$. A *unicycle* is a connected graph containing exactly one simple cycle. Show that the following are equivalent:

1. G is a unicycle.
2. There exists an edge e in G such that $G - e$ is a tree ($G - e$ denotes G with e deleted).
3. G is a connected graph where m, the number of edges, is equal to n, the number of vertices.

6 Trucking Dilemma (20 points)

A trucking company owns n trucks and employs n drivers. Each driver has a list of favorite rigs, and will not drive any truck he does not like. The company is interested in assigning trucks to drivers in such a way that all n drivers will agree to use all n trucks. Each driver can drive only one truck in an assignment, and drivers are intensely possessive and do not share trucks. Show that an assignment that is acceptable to all drivers is possible *if and only if* the following holds: for any $k \leq n$, in all subsets of k drivers, the drivers collectively favor at least k different rigs.

7 Extra credit (15 points)

In question 2, you were asked to show that every simple planar graph has a 6-coloring. In this problem, you can improve that result, reducing the number of colors to five.

- **(5 points)** Suppose you were to try using your proof of the existence of a 6-coloring for this purpose, changing every instance of "6" to "5". In what case would your inductive step fail?
- **(10 points)** Further modify your original proof to take care of this case (hint: try to recolor the graph when you run into problems).

8 Extra extra credit (∞ points)

DO NOT ATTEMPT THIS UNLESS YOU HAVE FINISHED THE REST OF THE ASSIGNMENT!

Luis has promised approximately one million points to any student who can prove that every simple planar graph has a 4-coloring. The proof must fit on at most five pages of 11pt text. The margins must be at least one inch in width. Proofs attempting to circumvent these requirements will not be accepted. No partial credit will be given for incorrect answers.