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Math 55
Homework #14
2005-10-31

Connection: Chapter 48 Problems 4, 5, 6, 9, 11

Discussed 48.4(c), 48.6, and 48.9 with Chris Roberts.

> **48.4** Let $n \geq 2$ be an integer. Form a graph G_n whose vertices are all the two-element subsets of $\{1, 2, \ldots, n\}$. In this graph we have an edge between distinct vertices $\{a, b\}$ and $\{c, d\}$ exactly when $\{a, b\} \cap \{c, d\} = \varnothing$.
> Please answer:
> (a) How many vertices does G_n have?
> (b) How many edges does G_n have?
> (c) For which values of $n \geq 2$ is G_n connected? Prove your answer.

Since the vertices of G_n are all of the 2-element subsets that can be chosen from an n-set, there are $\binom{n}{2}$ vertices in G_n.

$$|V(G_n)| = \binom{n}{2}$$

To count the number of edges in G_n, we proceed as follows: Each edge is incident upon 2 vertices, so it can be seen as incident upon 4 distinct elements of the source n-set. There are $\binom{n}{4}$ ways to choose these elements. Then, there are $\binom{4}{2} = 6$ ways we can select pairs of them, of which half are unique, giving 3 pairings among 4 elements. This means that for every distinct set of 4 elements in our source n-set, there are 3 edges in G_n. Therefore,

$$|E(G_n)| = 3\binom{n}{4}$$

Consider G_2. Since it has only $\binom{2}{2} = 1$ vertex, it is connected.

Now consider G_3. It has 3 vertices, and no edges, since every vertex shares an element with each of the other two. Therefore, it is not connected. G_4 is bipartite but not connected, since the vertices are paired.

Now consider an arbitrary graph G_n, with $n > 4$. Consider a vertex in this graph $\{a, b\}$. First, this vertex is trivially connected to itself. This vertex is also adjacent to, and therefore connected to any vertex composed of two other distinct elements $\{c, d\}$. Now, if this vertex is connected to all vertices that share an element, then G_n is connected. Without loss of generality, let us call this latter vertex $\{a, e\}$, with e being another distinct element. We can see that there is a path $\{a, b\}, \{c, d\}, \{a, e\}$ between these two vertices, with the only requirement being that a, b, c, d, and e be distinct. This implies that if we have at least 5 distinct elements from which to construct vertices, then any two vertices are connected by a path of length 2.

Therefore, G_n is connected for $n = 2$ and $n \geq 5$. ■

48.5 Consider the following (incorrect) restatement of the definition of connected: "A graph G is *connected* provided there is a path that contains every pair of vertices in G." What is wrong with this sentence?

Consider a star-shaped graph G with at least 4 vertices. If we call the center vertex c, then any two non-central vertices u, v are connected via a path (u, c, v), so G is connected. This is a maximal path in G, because the only possible next vertex would be a repetition of c. This means that a fourth distinct vertex w cannot appear in such a path. Thus, G is connected despite the fact that the pair of vertices cw cannot appear in such a path. ■

48.6 Let G be a graph. A path P in G that contains all the vertices of G is called a *Hamiltonian path*. Prove the the graph composed of a square grid of size 8×8 with two opposite corners missing does not have a Hamiltonian path.

Color the graph G in two colors such that no adjacent vertices have the same color. We can see that in constructing any path in G, we must move from one color to the other at each step, and so a path in G can contain at most 1 more vertex of one color than of the other. However, G has two more vertices of one color than than other, so it is not possible to construct a path in G that contains every vertex. ■

48.9 Let G be a graph. Prove that G or $\overline{G}$ (or both) must be connected.

Let G be a graph with a single vertex. Clearly, G is connected, because there is a path containing its single vertex. Additionally, $\overline{G}$ is connected as well.

Now, consider adding a vertex v to a graph H that satisfies the condition that H is connected or $\overline{H}$ is connected. Call the resultant graph I. Now, we wish to show that either I is connected or $\overline{I}$ is connected.

The simplest case to consider is where H is connected and v is adjacent to some vertex in H, making I connected. Similarly, if H was connected and v is not adjacent to any of the vertices in H, then $\overline{I}$ will have every node of H adjacent to v, and so it will be connected.

Now, consider the case in which H is not connected, but $\overline{H}$ is. If v is not adjacent to any vertices in $\overline{I}$, then it will be adjacent to all vertices in I, and so I will be connected. Finally, if v is adjacent to any vertex in $\overline{H}$, then $\overline{I}$ is connected.

Therefore, by the Principle of Mathematical Induction, either a graph or its complement must be connected.

■

48.11 Let G be a graph with $n \geq 2$ vertices.
(a) Prove that if G has at least $\binom{n-1}{2} + 1$ edges, then G is connected.
(b) Show that the result in (a) is best possible; i.e., for each $n \geq 2$, prove that there is a graph with $\binom{n-1}{2}$ edges that is not connected.

Consider an arbitrary graph G with $n \geq 2$ vertices and $\binom{n-1}{2} + 1$ edges. Suppose, for the sake of contradiction, that one of the vertices were not adjacent to any others. This leaves $n - 1$ vertices among which all of the edges must be distributed. This is impossible, since only $\binom{n-1}{2}$ edges can be drawn among these vertices. Therefore, there must be an edge joining the extra vertex to some vertex in the remaining connected component, and so G is connected.

Now consider a complete graph K_{n-1}, with $n \geq 2$. This has $\binom{n-1}{2}$ edges, because that is the number of ways to choose 2 vertices to be adjacent, and every vertex must be adjacent to all others. Now, construct a graph H with n vertices by adding a single vertex to K_{n-1} that is not adjacent to any other vertex. H is an example of a graph with n vertices and $\binom{n-1}{2}$ edges that is not connected. ∎