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Math 275

Assignment 1 - Sec 1.1 # 1.4, 1.6

1.4) Let $\Omega = \mathbb{R}, \mathcal{F} =$ all subsets so that A or A^c is countable, $P(A) = 0$ in the first case $= 1$ in the second. Show that $(\Omega, \mathcal{F}, P)$ is a probability space.

Pf : First we will show that $\mathcal{F}$ is a σ-algebra. $\emptyset \in \mathcal{F}$ since it is countable. Likewise $\mathbb{R} \in \mathcal{F}$ since $\mathbb{R}^c = \emptyset$ is countable. If $A \in \mathcal{F}$ then $A^c \in \mathcal{F}$ by definition. Finally if $A_1, A_2, ..., A_n, ... \in \mathcal{F}$ we have two cases. If A_i is countable for all i then $\bigcup A_i$ is countable thus in $\mathcal{F}$. If there exists an i such that A_i is not countable then $A_i^c \supset (\bigcup A_k)^c$. Thus $(\bigcup A_k)^c$ countable and $\bigcup A_i \in \mathcal{F}$.

Clearly $P(A) \geq 0$ for all $A \in \mathcal{F}$.

If $A_1, A_2, ..., A_n, ... \in \mathcal{F}$ is a sequence of disjoint sets we first note that for at most one i the set A_i^c is countable (since the complement of any countable set in $\mathbb{R}$ is uncountable. Further, if A_i^c is countable then A_i is uncountable.) So if A_i is countable for all i then $\bigcup A_i$ is countable so $P(\bigcup A_i) = 0 = \sum P(A_i)$. If there exists i such that A_i is uncountable then $P(\bigcup A_k) = 1 = 0 + P(A_i) = \sum_{k \neq i} P(A_k) + P(A_i) = \sum_k P(A_k)$.

And finally $P(\mathbb{R}) = 1$.

Therefore $(\Omega, \mathcal{F}, P)$ is a probability space.

1.6) Suppose X and Y are random variables on $(\Omega, \mathcal{F}, P)$ and let $A \in \mathcal{F}$. Show that if we let $Z(\omega) = X(\omega)$ for $\omega \in A$ and $Z(\omega) = Y(\omega)$ for $\omega \in A^c$, the Z is a random variable.
Pf : Let $B \subset \mathbb{R}$ be a Borel set. $Z^{-1}(B) = (X^{-1}(B) \cap A) \cup (Y^{-1}(B) \cap A^c) = (X^{-1}(B)^c \cup A^c)^c \cup (Y^{-1}(B)^c \cup A)^c$. But this last equality is just unions and complements of sets in $\mathcal{F}$ thus it is in $\mathcal{F}$. Therefore $Z^{-1}(B)$ is in $\mathcal{F}$.